

EXCEPTIONAL HOLONOMY AND EINSTEIN METRICS CONSTRUCTED FROM ALOFF-WALLACH SPACES

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ABSTRACT. We investigate cohomogeneity-one metrics whose principal orbit is an Aloff-Wallach space $SU(3)/U(1)$. In particular, we are interested in metrics whose holonomy is contained in $\text{Spin}(7)$. Complete metrics of this kind which are not product metrics have exactly one singular orbit. We prove classification results for metrics on tubular neighborhoods of various singular orbits. Since the equation for the holonomy reduction has only few explicit solutions, we make use of power series techniques. In order to prove the convergence and the smoothness near the singular orbit, we apply methods developed by Eschenburg and Wang. As a by-product of these methods, we find many new examples of Einstein metrics of cohomogeneity one.

1. INTRODUCTION

Metrics with holonomy $\text{Spin}(7)$ are an active area of research in differential geometry and in superstring theory (see Acharya, Gukov [1]). Most of the explicitly known examples (see Bazaikin [4], [5]; Bazaikin, Malkovich [6]; Bryant, Salamon [9]; Cvetič et al. [12], [13]; Gukov, Sparks [20]; Kanno, Yasui [23], [24]) are of cohomogeneity one. The advantage of cohomogeneity one metrics is that the equation for the holonomy reduction is equivalent to a system of first-order ordinary differential equations. Among those metrics, the metrics with an Aloff-Wallach space as the principal orbit are an interesting subclass.

An Aloff-Wallach space is a coset space $N^{k,l} := SU(3)/U(1)_{k,l}$ where $U(1)_{k,l} := \{\text{diag}(e^{kit}, e^{lit}, e^{-i(k+l)t}) | t \in \mathbb{R}\}$. The spaces $N^{1,0}$ and $N^{1,1}$ have a different geometry as the other ones and are called exceptional Aloff-Wallach spaces. Any principal orbit of a manifold carrying a parallel cohomogeneity-one $\text{Spin}(7)$ -structure, is equipped with a cocalibrated homogeneous G_2 -structure. We therefore will describe how a connected component of the space of all cocalibrated $SU(3)$ -invariant G_2 -structures on $N^{k,l}$ looks like. This problem can be solved by means of representation theory. After that we are able to deduce a system of ordinary differential equations which is

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equivalent to the holonomy reduction. For reasons of simplicity, we carry out this program only for metrics which are diagonal with respect to a certain basis.

If a cohomogeneity-one metric with holonomy $\text{Spin}(7)$ or a smaller group is complete and not a product, it has exactly one singular orbit. We therefore fix the initial values of our differential equations at the singular orbit. Unfortunately, these initial value problems have an explicit solution only in some special cases [4], [6], [13], [20]. Furthermore, the equations for the holonomy reduction degenerate near the singular orbit. More precisely, some of the summands behave like $\frac{0}{0}$. We therefore cannot apply the theorem of Picard-Lindelöf. In the literature [13], [23], [24] there are indeed examples, where the solutions do not only depend on the metric on the singular orbit but also on an initial condition of second or third order which can be chosen freely.

In order to solve these problems, we make a power series ansatz for the metric. We have to check if the power series converges and how many free parameters of higher order there are. Another problem is that not any solution of the differential equations corresponds to a metric which can be smoothly extended to the singular orbit. There are certain smoothness conditions which have to be satisfied and are in some cases a serious obstacle. If for example the principal orbit is $N^{1,1}$ and the singular orbit is $SU(3)/U(1)^2$, we have to replace the principal orbit by a quotient $N^{1,1}/\mathbb{Z}_2$ in order to satisfy the smoothness conditions.

The above problems were addressed by Eschenburg and Wang [15] in the context of cohomogeneity-one Einstein metrics. Although metrics with exceptional holonomy are Ricci-flat and thus Einstein metrics, we have to adapt the methods of [15] to our situation. After that we are finally able to prove the existence of metrics whose holonomy is a subgroup of $\text{Spin}(7)$ on a tubular neighborhood of the singular orbit. At this point we are able to apply the main theorem of Eschenburg and Wang [15] and can also show the existence of Einstein metrics of cohomogeneity one. With the exception of an $SU(3)$ -invariant Einstein metric on $\mathbb{H}\mathbb{P}^2$ which can be found in Püttmann, Rigas [27], all of the Einstein metrics are to the best knowledge of the author new.

The main results of the article can be summarized as follows. We find a two-parameter family of smooth non-homothetic cohomogeneity-one metrics with holonomy a subgroup of $\text{Spin}(7)$. All of them have $N^{1,1}/\mathbb{Z}_2$ as principal orbit and $SU(3)/U(1)^2$ as singular orbit. Among them, there is a one-parameter family of metrics with holonomy $SU(4)$. At the border of the moduli space the metric converges to the Calabi metric (see [10]) on $T^*\mathbb{CP}^2$, which has holonomy $Sp(2)$. First evidence for the above metrics can be found in Kanno, Yasui [24]. The two-parameter family and the metrics with

holonomy $SU(4)$ were investigated independently of the author by Bazaikin and Malkovich [4], [6].

In [13], [23] and [24], metrics with holonomy $\text{Spin}(7)$ are constructed by numerical methods. One family of metrics has principal orbit $N^{1,0}$ and singular orbit S^5 . Another family has an arbitrary $N^{k,l}$ with $(k,l) \neq (1,-1)$ as principal orbit and \mathbb{CP}^2 as singular orbit. These are the metrics from [13] and [23] which depend on a free parameter of third order and which we have already mentioned above. The space on which the metrics are defined is an $\mathbb{R}^4/\mathbb{Z}_{|k+l|}$ -bundle over \mathbb{CP}^2 and thus an orbifold. If the principal orbit is the exceptional Aloff-Wallach space $N^{1,1}$, there are further metrics with singular orbit \mathbb{CP}^2 which are not of the above kind. Examples of these metrics can be found in [24]. There exists a further two-parameter family of non-homothetic metrics with that orbit structure which was found by Bazaikin [5]. We also construct this metrics and show that the family can be parameterized by two free parameters of third order.

Moreover, we prove for all of our cases with the help of the methods of Eschenburg and Wang [15] that there are no other free parameters except the known ones, that the smoothness conditions are satisfied and that the power series solutions converge. We also prove that under certain conditions, for example that the metric is diagonal, there are no further cohomogeneity one metrics whose holonomy is contained in $\text{Spin}(7)$. Finally, we prove that the holonomy of the metrics whose principal orbit is not $N^{1,1}$ is all of $\text{Spin}(7)$.

The article is organized as follows. In the second section, we collect some basic facts on G_2 - and $\text{Spin}(7)$ -structures. In Section 3, cohomogeneity-one manifolds and the methods of Eschenburg and Wang [15] are introduced. The fourth section deals with the geometry of the Aloff-Wallach spaces. Our metrics with cohomogeneity one are constructed in the remaining three sections. In each section, we deal with metrics which have one particular singular orbit. The fifth section is about metrics with singular orbit $SU(3)/U(1)^2$, the sixth about S^5 , and the final section is about metrics with \mathbb{CP}^2 as singular orbit.

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2. G_2 - AND $\text{SPIN}(7)$ -STRUCTURES

In this section, we define some terms which we will often use later on. Let (dx^1, \dots, dx^7) be the standard basis of one-forms on \mathbb{R}^7 . Furthermore, let

$$(1) \quad \omega := dx^{123} + dx^{145} - dx^{167} + dx^{246} + dx^{257} + dx^{347} - dx^{356},$$

where $dx^{i_1 i_2 \dots i_k}$ denotes $dx^{i_1} \wedge \dots \wedge dx^{i_k}$. The Hodge dual of ω is

$$(2) \quad *\omega = -dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}.$$

We supplement (dx^1, \dots, dx^7) with dx^0 to a basis of one-forms on \mathbb{R}^8 and define

$$(3) \quad \begin{aligned} \Omega &:= *\omega + dx^0 \wedge \omega \\ &= dx^{0123} + dx^{0145} - dx^{0167} + dx^{0246} + dx^{0257} + dx^{0347} - dx^{0356} \\ &\quad - dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}. \end{aligned}$$

A G_2 -structure on a seven-dimensional manifold is a three-form which can be identified via local frames with ω . Analogously, a $Spin(7)$ -structure on an eight-dimensional manifold is a four-form which can be identified at each point with Ω . To any G_2 -structure or $Spin(7)$ -structure we can associate a canonical metric g and an orientation. We call a G_2 -structure or $Spin(7)$ -structure ω or Ω *parallel* if $\nabla^g \omega = 0$ or $\nabla^g \Omega = 0$. A pair (N, ω) or (M, Ω) of a seven-dimensional or eight-dimensional manifold and a parallel G_2 -structure or $Spin(7)$ -structure is called a G_2 -manifold or $Spin(7)$ -manifold. Those manifolds have the following interesting properties.

Theorem 2.1. (See Bonan [7]; Fernández, Gray [16]; Fernández [17])

- (1) Let N be a seven-dimensional manifold with a G_2 -structure ω . Then, the following statements on ω are equivalent.
 - (a) ω is parallel.
 - (b) $d\omega = d*\omega = 0$.
 - (c) The holonomy of the associated metric g is contained in G_2 .

Conversely, if (N, g) is a Riemannian manifold with holonomy a subgroup of G_2 , then there exists a parallel G_2 -structure on N such that its associated metric is g . If any of the above conditions is satisfied, g is Ricci-flat.
- (2) Let M be an eight-dimensional manifold with a $Spin(7)$ -structure Ω . Then, the following statements on Ω are equivalent.
 - (a) Ω is parallel.
 - (b) $d\Omega = 0$.
 - (c) The holonomy of the associated metric g is contained in $Spin(7)$.

Conversely, if (M, g) is a Riemannian manifold with holonomy a subgroup of $Spin(7)$, then there exists a parallel $Spin(7)$ -structure on N such that its associated metric is g . If any of the above conditions is satisfied, g is Ricci-flat.

Finally, we introduce the following types of non-parallel G_2 -structures which we will need later on.

Definition 2.2. A G_2 -structure ω is called

- (1) *nearly parallel* if $d\omega = \lambda * \omega$ for a $\lambda \in \mathbb{R} \setminus \{0\}$.
- (2) *cocalibrated* if $d * \omega = 0$.

3. COHOMOGENEITY-ONE MANIFOLDS

The set of all $\text{Spin}(7)$ -structures on a manifold M does not define a vector subbundle of $\bigwedge^4 T^*M$. Therefore, the condition $d\Omega = 0$ should be considered as a non-linear partial differential equation. Explicit solutions of this equation are hard to find. Among them are the first examples of complete metrics with holonomy $\text{Spin}(7)$ by Bryant and Salamon [9]. The existence of the non-explicit metrics of Bryant [8] and of Joyce [22] can only be proven by sophisticated analytic arguments. Our problem becomes a lot of simpler if we assume that Ω is of cohomogeneity one.

Definition and Lemma 3.1. (Cf. Mostert [26] and references therein)

- (1) Let M be an n -dimensional connected manifold with a smooth action by a Lie group G . The action of G is called a *cohomogeneity-one action* if there exists an orbit with dimension $n - 1$.
- (2) An orbit \mathcal{O} of a cohomogeneity-one action is called a *principal orbit* if there is an open subset U of M with the following properties: $\mathcal{O} \subseteq U$ and U is G -equivariantly diffeomorphic to $\mathcal{O} \times (-\epsilon, \epsilon)$, where $\epsilon > 0$. It can be proven that this condition is equivalent to $\dim \mathcal{O} = n - 1$. All principal orbits are G -equivariantly diffeomorphic to each other and the union of all principal orbits is an open dense subset of M .
- (3) A $\text{Spin}(7)$ -manifold (M, Ω) is called *of cohomogeneity one* if there exists a cohomogeneity-one action on M which preserves Ω (and thus the associated metric).

Since any Ricci-flat homogeneous metric is flat (see Alekseevskii, Kimelfeld [2]), spaces of cohomogeneity one are the most symmetric manifolds which may admit metrics with exceptional holonomy. For the following considerations, we fix some notation.

Convention 3.2. Let G be a compact Lie group which acts with cohomogeneity one on a $\text{Spin}(7)$ -manifold (M, Ω) . The associated metric on M we denote by g . We identify any orbit of G with the quotient of G by the isotropy group. The principal orbit shall be G/H and G/K shall be a non-principal orbit. The union of all principal orbits will be denoted by M^0 . After conjugation, H has to be a subgroup of K . The Lie algebras of G , H , and K , we denote by \mathfrak{g} , \mathfrak{h} , and \mathfrak{k} . Let q be an auxiliary Ad_K -invariant metric on \mathfrak{g} . We identify the tangent space of G/H with the q -orthogonal complement \mathfrak{m} of \mathfrak{h} in \mathfrak{g} . The tangent space of G/K can be identified with the complement \mathfrak{p} of \mathfrak{k} . We denote the normal space of the orbit G/K by \mathfrak{p}^\perp . On any cohomogeneity-one manifold, there exists a geodesic which intersects

all orbits perpendicularly. We fix such a geodesic γ and parameterize it by arclength. The parameter of γ we denote by t .

The following theorem of Mostert [26] gives us some information on the shape of M .

Theorem 3.3. (1) *If G acts isometrically on a Riemannian cohomogeneity-one manifold (M, g) , M/G is homeomorphic to the circle S^1 , $[0, 1]$, $[0, \infty)$, or \mathbb{R} . The inner points of M/G correspond to principal orbits and the endpoints of the intervals to non-principal orbits.*

(2) *Let G/H be a principal and G/K be a non-principal orbit. The quotient K/H is a sphere.*

(3) *Any sufficiently small tubular parameterize of the non-principal orbit G/K is a disc bundle over G/K . The projection map maps a point gH of the principal orbit to gK .*

Remark 3.4. Spin(7)-manifolds with certain kinds of singularities are in issue in M-theory (see Acharya, Gukov [1]). Therefore, we also consider the cases where M is not a manifold but an orbifold. If K/H is a quotient of a sphere by a discrete group Γ , the tubular parameterize of G/K is an $\mathbb{R}^{\dim K/H+1}/\Gamma$ -bundle over G/K and M thus is an orbifold. In this section, we state our theorems for manifolds only. Nevertheless, it is easily possible to adapt them to the orbifold-case.

Since the volume of the metric on K/H shrinks to zero as we approach the singular orbit, we will refer to K/H as the *collapsing sphere* or if $\dim K/H = 1$, as the *collapsing circle*. We can restrict the topology of M even further. If $M/G = \mathbb{R}$, (M, g) contains a complete geodesic which minimizes the length between any of its points. It follows from the Cheeger-Gromoll splitting theorem that M is a Riemannian product of \mathbb{R} and a seven-dimensional manifold. Since then the holonomy would be a subgroup of G_2 , we will not consider this case. If M/G was S^1 , the universal cover \widetilde{M} would satisfy $\widetilde{M}/G = \mathbb{R}$. Therefore, we exclude that case, too. If M had two non-principal orbits, it would be compact. Since it is Ricci-flat, all Killing vector fields are parallel and commute with each other. G/H thus is a flat torus. It is easy to see that M has to be flat, too.

There are two kinds of non-principal orbits. If $K/H = S^0 = \mathbb{Z}_2$, the orbit G/K is called an *exceptional orbit*. Otherwise, it is a *singular orbit*. If there is exactly one exceptional orbit, M would be twofold covered by a space \widetilde{M} with $\widetilde{M}/G = \mathbb{R}$. Motivated by the above considerations, we assume from now on that there is exactly one singular orbit and all other orbits are principal.

On any principal orbit, there exists a canonical G_2 -structure ω which is related to Ω by $\Omega := *\omega + dt \wedge \omega$. The equation $d\Omega = 0$ can be written in terms of ω .

Theorem 3.5. *Let G/H be a seven-dimensional homogeneous space and ω be a G -invariant cocalibrated G_2 -structure on G/H . Then there exists an $\epsilon > 0$ and a one-parameter family $(\omega_t)_{t \in (-\epsilon, \epsilon)}$ of G -invariant G_2 -structures on G/H such that the initial value problem*

$$(4) \quad \frac{\partial}{\partial t} *_{G/H} \omega_t = d_{G/H} \omega_t$$

$$(5) \quad \omega_0 = \omega$$

has a unique solution on $G/H \times (-\epsilon, \epsilon)$. In the above formula, $\frac{\partial}{\partial t}$ denotes the Lie derivative in t -direction. The index G/H of d and $*$ emphasizes that we consider the exterior derivative on G/H instead of $G/H \times (-\epsilon, \epsilon)$. If ϵ is sufficiently small, ω_t is for all $t \in (-\epsilon, \epsilon)$ a G_2 -structure and we have $d_{G/H} *_{G/H} \omega_t = 0$. The four-form $\Omega := *_{G/H} \omega + dt \wedge \omega$ is a G -invariant parallel $\text{Spin}(7)$ -structure on $G/H \times (-\epsilon, \epsilon)$.

Conversely, let Ω be a parallel $\text{Spin}(7)$ -structure preserved by a cohomogeneity-one action of a Lie group G . We identify the union of all principal orbits G -equivariantly with $G/H \times I$, where the metric on I is dt^2 . In this situation, the G_2 -structures on the principal orbits are cocalibrated and satisfy equation (4).

- Remark 3.6.* (1) The above theorem was proven by Hitchin [21] for the more general case, where ω is a (not necessarily homogeneous) cocalibrated G_2 -structure on a compact manifold.
- (2) If ω is nearly parallel, the maximal solution of (4) describes a cone over G/H .
- (3) Since (M, Ω) is of cohomogeneity one, the equation (4) is equivalent to a system of ordinary differential equations. In order to ensure that M has the desired topology, we fix the initial conditions at the singular orbit G/K .

Before we investigate the equation (4), we have to choose the principal orbit. The following lemma answers the question if G/H admits a G -invariant G_2 -structure.

Lemma 3.7. (See [29].) *Let G/H be a homogeneous space such that G acts effectively on G/H . Furthermore, let $p \in G/H$ be arbitrary. We identify H with its isotropy representation on $T_p G/H$ and G_2 with its seven-dimensional irreducible representation. G/H admits a G -invariant G_2 -structure if and only if there exists a vector space isomorphism $\varphi : T_p G/H \rightarrow \mathbb{R}^7$ such that $\varphi H \varphi^{-1} \subseteq G_2$.*

Our next step is to describe the space of all G -invariant G_2 -structures on G/H explicitly. This can be done in two steps. Any G -invariant metric on G/H can be identified via q with an H -equivariant endomorphism of \mathfrak{m} . These endomorphisms can be classified with the help of Schur's lemma.

Since any manifold which admits a G_2 -structure is orientable and an $SO(7)$ -structure is the same as a metric and an orientation, we have classified all G -invariant $SO(7)$ -structures on G/H . The classification of all G -invariant G_2 -structures whose extension to an $SO(7)$ -structure is fixed, can be done with the help of the following lemma.

Lemma 3.8. *(See [30].) Let G/H be a seven-dimensional homogeneous space. We assume that G acts effectively and that G/H admits a G -invariant G_2 -structure. Let \mathcal{G} be an arbitrary G -invariant $SO(7)$ -structure on G/H . The space of all G -invariant G_2 -structures on G/H whose extension to an $SO(7)$ -structure is \mathcal{G} is diffeomorphic to*

$$(6) \quad \text{Norm}_{SO(7)} H / \text{Norm}_{G_2} H .$$

In the above formula, H is identified with its isotropy representation and G_2 and $SO(7)$ with their seven-dimensional irreducible representation. The normalizer $\text{Norm}_L L'$ of a subgroup $L' \subseteq L$ is defined as $\{g \in L | gL'g^{-1} = L'\}$.

After having determined the space of all G -invariant G_2 -structures ω on G/H , we calculate $d*\omega$ and have a description of the space of all cocalibrated invariant G_2 -structures. In some cases, we will not be able to describe that space explicitly. In order to find examples of parallel cohomogeneity-one $\text{Spin}(7)$ -structures, it suffices to construct a space of cocalibrated G_2 -structures which is invariant under equation (4).

Not any solution of (4) corresponds to a metric with holonomy $\text{Spin}(7)$. The reason for this is that Ω has not automatically a smooth extension to the singular orbit. This is the case only if certain smoothness conditions are satisfied, which we will describe in detail.

We split the tangent space of M at a point $p \in G/K$ into the K -modules \mathfrak{p} and \mathfrak{p}^\perp . The orbits of the K -action on \mathfrak{p}^\perp except $\{0\}$ are spheres of type K/H . Let \mathcal{B} be a vector bundle over the union of all principal orbits which admits a K -action on the fibers. For reasons of simplicity we assume that there exist non-negative numbers s_1 and s_2 such that the fibers of \mathcal{B} are contained in $\bigotimes^{s_1} TM \otimes \bigotimes^{s_2} T^*M$. Moreover, let ρ be a G -invariant section of \mathcal{B} . Since G/K is homogeneous, ρ is determined by its values at p .

Let γ be a geodesic which intersects all orbits perpendicularly. We assume that $\gamma(0) \in G/K$. Since the action of G on γ generates all of M , it suffices to consider ρ along γ only. The metric g is Ricci-flat. It is well-known (see DeTurck, Kazdan [14]) that any Einstein metric is analytic. We therefore assume that ρ is a power series with respect to t . The m^{th} derivative of ρ in the vertical direction can be considered as a map, which assigns to a tuple $(v_1, \dots, v_m) \in \mathfrak{p}^\perp$ an element of the fiber \mathcal{B}_p . This map can be extended to a map $S^m(\mathfrak{p}^\perp) \rightarrow \mathcal{B}_p$, where $S^m(\mathfrak{p}^\perp)$ denotes the m^{th} symmetric power of \mathfrak{p}^\perp . Since ρ is analytic, the sequence of those maps determines ρ . If ρ has a

smooth extension to the singular orbit, the above maps are K -equivariant. Conversely, we have

Theorem 3.9. (See Eschenburg, Wang [15].) *Let (M, g) be Riemannian manifold with an isometric action of cohomogeneity one by a Lie group G . We assume that there is a singular orbit G/K . Let $\mathcal{B} \subseteq \bigotimes^{s_1} TM \otimes \bigotimes^{s_2} T^*M$ be a vector bundle over M whose fibers at the singular orbit are K -equivariantly isomorphic to a K -module B . Let $r : (0, \varepsilon) \rightarrow B$, where $\varepsilon > 0$, be a real analytic map with Taylor expansion $\sum_{m=1}^{\infty} r_m t^m$. We can identify r with a tensor field ρ along a geodesic γ which intersects all orbits perpendicularly. By the action of G , we can extend ρ to the union of all principal orbits. ρ is well-defined and has a smooth extension to the singular orbit if and only if*

$$(7) \quad r_m \in \imath_m(W_m) \quad \forall m \in \mathbb{N}_0.$$

In the above formula, W_m denotes the space of all K -equivariant maps

$$(8) \quad W_m := \{P : S^m(\mathfrak{p}^\perp) \rightarrow B \mid P \text{ is linear and } K\text{-equivariant}\}$$

and \imath_m is the evaluation map

$$(9) \quad \begin{aligned} \imath_m : W_m &\rightarrow B \\ \imath_m(P) &:= P(\gamma'(0)). \end{aligned}$$

In the following, we restrict ourselves to metrics with no "mixed coefficients", i.e.

$$(10) \quad g \in S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp).$$

Instead of W_m , it suffices to study the spaces

$$(11) \quad \begin{aligned} W_m^h &:= \{P : S^m(\mathfrak{p}^\perp) \rightarrow S^2(\mathfrak{p}) \mid P \text{ is linear and } K\text{-equivariant}\} \quad \text{and} \\ W_m^v &:= \{P : S^m(\mathfrak{p}^\perp) \rightarrow S^2(\mathfrak{p}^\perp) \mid P \text{ is linear and } K\text{-equivariant}\} \end{aligned}$$

in order to prove the smoothness. The reason for assumption (10) is that later on we need a result which is proven only if (10) is satisfied.

We often write our metric as $g_t + dt^2$ where $g_t \in S^2(\mathfrak{m})$ is the restriction of g to a principal orbit. Since t can be considered as a distance function on \mathfrak{p}^\perp and $g_t|_{\mathfrak{p}^\perp \times \mathfrak{p}^\perp}$ describes the metric on the collapsing sphere, \mathfrak{p}^\perp is equipped with "polar" rather than "Euclidean" coordinates. We therefore have to modify Theorem 3.9 in order to fit our needs.

- Remark 3.10.* (1) By the choice of our coordinates we have fixed $\|\frac{\partial}{\partial t}\| = 1$ and $g(\frac{\partial}{\partial t}, v) = 0$ for all $v \in \mathfrak{m}$. The degrees of freedom for the higher derivatives of the vertical part of g will therefore seem to be fewer as Theorem 3.9 predicts.
- (2) Let v be a tangent vector of the collapsing sphere K/H . The metric on K/H has to approach the round metric of a sphere of radius t . This condition fixes the value of $\frac{\partial}{\partial t}|_{t=0}g_t(v, v)^{\frac{1}{2}}$. If K/H is a sphere, we can compute this value with the help of the fact that the length of any great circle on K/H has to be $2\pi t + O(t^2)$ for $t \rightarrow 0$. If K/H is a quotient of a sphere by a discrete group, we can use the estimate $\frac{1}{t} + O(1)$ for the sectional curvature. The above statements are in fact equivalent to the smoothness condition of 0^{th} order for the vertical part.
- (3) Since the length of v shrinks to zero, any statement on the m^{th} derivative of g in the vertical direction translates into a statement on $\frac{\partial^m}{\partial t^m}|_{t=0}\frac{1}{t}g_t(v, v)^{\frac{1}{2}}$. Because of l'Hôpital's rule this is essentially a statement on the $(m+1)^{st}$ derivative of g_t .

We assume that (M^0, Ω) is of holonomy $\text{Spin}(7)$ and that the metric g has a smooth extension to the singular orbit. In this situation, the holonomy of (M, g) equals $\text{Spin}(7)$, too. Therefore, there exists a unique smooth $\text{Spin}(7)$ -structure $\tilde{\Omega}$ on M . Without loss of generality, we can assume that Ω and $\tilde{\Omega}$ coincide on M^0 . This observation proves that $\tilde{\Omega}$ is a smooth extension of Ω to the singular orbit and we do not have to prove the smoothness conditions for Ω .

If the holonomy Hol is a smaller group, for example $Sp(2)$ or $SU(4)$, we can prove by similar arguments that there exists a smooth Hol -structure on M . Since G acts by isometries, it leaves the holonomy bundle invariant and the Hol -structure thus is G -invariant.

Since $\dim G/K < \dim G/H$, the equation (4) sometimes degenerates at the singular orbit. More precisely, it is equivalent to a system which contains equations of type $c'(t) = \dots + \frac{a(t)}{b(t)} + \dots$ with $\lim_{t \rightarrow 0} a(t) = \lim_{t \rightarrow 0} b(t) = 0$. In that situation, we cannot apply the theorem of Picard-Lindelöf, since the right-hand side $f(a, b, c, \dots)$ is not defined on an open set. There are indeed cases, where the solution of our initial value problem depends on initial conditions of higher order which can be chosen freely. In order to classify the solutions of (4), we make the power series ansatz

$$(12) \quad \omega = \sum_{m=0}^{\infty} \omega_m t^m \quad \text{with} \quad \omega_m \in \bigwedge^3 \mathfrak{m} \quad \text{Ad}_H\text{-invariant}.$$

In the cases which we will consider, we fix for any choice of the metric g_t on G/H a single cocalibrated G_2 -structure ω_t whose associated metric is g_t . The cocalibrated G_2 -structures with that property are in many cases a discrete set. The set of G_2 -structures which is obtained from a sufficiently large set of g_t is preserved by (4) and our restriction to those G_2 -structures thus is justified. ω_t will always depend analytically on g_t and the cohomogeneity-one metric g is Ricci-flat. Since any Einstein metric is analytic (see DeTurck, Kazdan [14]), we are allowed to make the above power series ansatz. Equation (4) yields the following system of recursive equations for the g_m

$$(13) \quad \mathcal{L}_m(g_m) = P_m(g_0, \dots, g_{m-1}).$$

\mathcal{L}_m is a linear operator acting on the space of all Ad_H -invariant symmetric bilinear forms on \mathfrak{m} and P_m is a polynomial. For a fixed choice of the principal and the singular orbit, \mathcal{L}_m can be calculated for all m . We will see that in each of our cases there exists an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ \mathcal{L}_m is always invertible. It follows from Theorem 3.12, which we will state below, that there is a deeper reason behind this. By solving (13) for all $m < m_0$, we can classify all formal power series which solve equation (4). By certain arguments which we will make explicit when we need them we can check the smoothness conditions. All which is left to be done is to check if the power series converges. This follows by a theorem of Eschenburg and Wang [15] on cohomogeneity-one Einstein metrics. Before we state that theorem, we make the following assumption.

Assumption 3.11. The tangent space \mathfrak{p} and the normal space \mathfrak{p}^\perp of the singular orbit shall have no H -submodule of positive dimension in common.

Theorem 3.12. (See Eschenburg, Wang [15].) *Let M be a manifold equipped with a cohomogeneity-one action by a compact Lie group G . We assume that the principal orbits of this action are G -equivariantly diffeomorphic to G/H and that there is a singular orbit G/K . Moreover, we assume that 3.11 is satisfied.*

Let g_0 be an arbitrary G -invariant metric on the singular orbit. Furthermore, let $g'_0 : \mathfrak{p}^\perp \rightarrow S^2(\mathfrak{p})$ be a linear, K -equivariant map. Finally, let $\lambda \in \mathbb{R}$ be arbitrary. In this situation, there exists a G -invariant Einstein metric g on a sufficiently small tubular neighborhood of the singular orbit which has the following properties.

- (1) g has λ as Einstein constant.
- (2) The restriction of g to the singular orbit is g_0 .
- (3) The first derivation of g at the singular orbit in the normal directions is g'_0 .

The set of all Einstein metrics with the above properties depends on additional initial conditions of higher order, which we can prescribe arbitrarily.

The freedom for the m^{th} derivative of the metric in the horizontal or vertical direction can be described by

$$(14) \quad \begin{array}{ll} W_m^h/W_{m-2}^h & \text{in the horizontal case if } m \geq 2 \\ W_2^v/W_0^v & \text{in the vertical case if } m = 2 \end{array}$$

and there are no further free parameters in the vertical direction.

- Remark 3.13.* (1) Up to constant multiples, there is exactly one K -invariant scalar product h on \mathfrak{p}^\perp . The reason for this is that the orbits of the K -action on \mathfrak{p}^\perp are spheres. In particular, we have $\dim W_0^v = 1$.
- (2) $S^m(\mathfrak{p}^\perp)$ is embedded into $S^{m+2}(\mathfrak{p}^\perp)$ by the map ι with $\iota(P) := h \vee P$, where \vee is the symmetrized tensor product. ι induces canonical embeddings of W_{m-2}^h and W_{m-2}^v into W_m^h and W_m^v , which we have implicitly used in the formulation of the above theorem.
- (3) Since we assume that 3.11 is satisfied, the metric is automatically contained in $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. Analogously, we have $W_m = W_m^h \oplus W_m^v$.
- (4) For sufficiently large m the chains $W_0 \subseteq W_2 \subseteq W_4 \subseteq \dots$ and $W_1 \subseteq W_3 \subseteq W_5 \subseteq \dots$ stabilize. In particular, there are only finitely many initial conditions which we can prescribe.
- (5) The cohomogeneity-one Einstein condition is system of second order differential equations. That system yields a recursive equation which is similar to (13). The convergence of the power series solutions can be shown with the help of a Picard iteration. More precisely, any power series which satisfies the Einstein condition converges if \mathfrak{p} and \mathfrak{p}^\perp have no H -submodule in common. In particular, any formal power series solution of (13) converges if we assume 3.11. In some of the cases which we will consider, this assumption is not satisfied. Nevertheless, the convergence can be proven in those cases, too. In the article, we restrict ourselves to metrics which are diagonal with respect to a fixed basis of \mathfrak{m} . The metric is therefore an element of $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. Moreover, the equations for the Einstein condition do not change a diagonal metric into a non-diagonal one. The space $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$ thus is invariant under the Picard iteration. This allows us to repeat the arguments of [15] and to prove the convergence.
- (6) In order to deduce the smoothness conditions, we have to describe the spaces W_m^h and W_2^v . We can therefore easily apply Theorem 3.12 and obtain new examples of cohomogeneity-one Einstein metrics.
- (7) The above theorem predicts that certain second derivatives in the vertical direction can be chosen freely. For similar reasons as in Remark 3.10, these derivatives are third derivatives with respect to t . As we have already remarked in 3.10, we have $g(\frac{\partial}{\partial t}, v) = 0$ for

all $v \in \mathfrak{m}$ by the choice of our coordinates. We therefore have to ignore the free parameters in the vertical direction which describe the change of $g(\frac{\partial}{\partial t}, v)$.

- (8) In general, the power series converges for small values of t only. The metrics which we construct with the help of the above theorem are thus incomplete. They can be extended to complete metrics if and only if the equation $\frac{\partial}{\partial t} * \omega = d\omega$ or $\text{Ric} = \lambda g$ has a solution for all $t \in [0, \infty)$ or there are two singular orbits.

After we have proven the convergence, we have constructed metrics whose holonomy group is a subgroup of $\text{Spin}(7)$ on a tubular parameterize of the singular orbit. In order to decide if the holonomy is all of $\text{Spin}(7)$, we need the following lemma.

Lemma 3.14. *(See [30].)*

- (1) *Let M be an eight-dimensional manifold which carries a parallel $SU(4)$ -structure \mathfrak{G} . We denote the space of all parallel $\text{Spin}(7)$ -structures on M which are an extension of \mathfrak{G} and have the same extension to an $SO(8)$ -structure as \mathfrak{G} by \mathcal{S} . Any connected component of \mathcal{S} is diffeomorphic to a circle.*
- (2) *Let M be an eight-dimensional manifold which carries a one-parameter family \mathcal{S} of parallel $\text{Spin}(7)$ -structures. Moreover, let the extension of the $\text{Spin}(7)$ -structures to an $SO(8)$ -structure always be the same and let \mathcal{S} be diffeomorphic to a circle. Then, there also exists a parallel $SU(4)$ -structure on M .*

With the help of the facts which we have collected in this section we are now able to construct examples of $\text{Spin}(7)$ -manifolds.

4. THE GEOMETRY OF THE ALOFF-WALLACH SPACES

Before we construct cohomogeneity-one metrics with an Aloff-Wallach space as the principal orbit, we have to study those spaces in detail. The Aloff-Wallach spaces are certain homogeneous spaces which were introduced by Aloff and Wallach [3] in order to study metrics with positive sectional curvature. Let

$$(15) \quad \begin{aligned} & i_{k,l} : U(1) \rightarrow SU(3) \\ & i_{k,l}(e^{i\varphi}) := \begin{pmatrix} e^{ik\varphi} & 0 & 0 \\ 0 & e^{il\varphi} & 0 \\ 0 & 0 & e^{-i(k+l)\varphi} \end{pmatrix} \quad \text{with } k, l \in \mathbb{Z}. \end{aligned}$$

We denote the image of $U(1)$ with respect to $i_{k,l}$ by $U(1)_{k,l}$. Any one-dimensional subgroup of $SU(3)$ is conjugate to a $U(1)_{k,l}$. The Aloff-Wallach

space $N^{k,l}$ is defined as the quotient $SU(3)/U(1)_{k,l}$. Without loss of generality, we assume that k and l are coprime. Let σ be a permutation of the triple $(k, l, -k-l)$. It is easy to see that the spaces $N^{k,l}$ and $N^{\sigma(k),\sigma(l)}$ are $SU(3)$ -equivariantly diffeomorphic. Let $2\mathfrak{u}(1)$ be the Cartan subalgebra of all diagonal matrices in $\mathfrak{su}(3)$ and let $\mathfrak{u}(1)_{k,l}$ be the Lie algebra of $U(1)_{k,l}$. The Weyl group of $\mathfrak{su}(3)$ is isomorphic to the permutation group S_3 and acts on $2\mathfrak{u}(1)$. The action of a $\sigma \in S_3$ on $2\mathfrak{u}(1)$ changes $\mathfrak{u}(1)_{k,l}$ into $\mathfrak{u}(1)_{\sigma(k),\sigma(l)}$. Therefore, the S_3 -action on $(k, l, -k-l)$ can be identified with the action of the Weyl group. Since $N^{k,l}$ and $N^{-k,-l}$ are the same manifold, we introduce the following convention.

Convention 4.1. When we consider an Aloff-Wallach space $N^{k,l}$, we assume that $k \geq l \geq 0$. Later on, we will turn to another convention which will be introduced at that point.

The Aloff-Wallach spaces $N^{1,0}$ or $N^{1,1}$ are called *exceptional* and the other ones are called *generic*. Since there are many differences between the exceptional and the generic Aloff-Wallach spaces, we often have to treat $N^{1,0}$, $N^{1,1}$, and the generic $N^{k,l}$ as separate cases. There are infinitely many homotopy types of Aloff-Wallach spaces. This follows from the fact that

$$(16) \quad H^4(N^{k,l}, \mathbb{Z}) = \mathbb{Z}_{k^2+lk+l^2}.$$

Some of the $N^{k,l}$ are homeomorphic but not diffeomorphic to each other. Examples of this fact can be found in Kreck, Stolz [25]. On the Aloff-Wallach spaces $N^{k,l}$ there exist two nearly parallel G_2 -structures, which depend on k and l [13]. The holonomy of the cones over those G_2 -structures therefore is contained in $\text{Spin}(7)$. We will see that the Aloff-Wallach spaces are not covered by a sphere. Therefore, the cones have a singularity at the tip, which is not an orbifold singularity.

Our next step is to describe all $SU(3)$ -invariant metrics on the Aloff-Wallach spaces. In order to do this, we fix the following basis $(e_i)_{1 \leq i \leq 8}$ of $\mathfrak{su}(3)$.

$$(17) \quad \begin{aligned} e_1 &:= E_1^2 - E_2^1 & e_2 &:= iE_1^2 + iE_2^1 & e_3 &:= E_1^3 - E_3^1 \\ e_4 &:= iE_1^3 + iE_3^1 & e_5 &:= E_2^3 - E_3^2 & e_6 &:= iE_2^3 + iE_3^2 \\ e_7 &:= (2l+k)iE_1^1 + (-2k-l)iE_2^2 + (k-l)iE_3^3 \\ e_8 &:= kiE_1^1 + liE_2^2 - (k+l)iE_3^3 \end{aligned}$$

E_i^j denotes the 3×3 -matrix with a 1 in the i^{th} row and j^{th} column and zeroes elsewhere. The Lie algebra $\mathfrak{u}(1)_{k,l}$ is generated by e_8 . $q(X, Y) := -\text{tr}(XY)$

defines a biinvariant metric on $\mathfrak{su}(3)$ and (e_1, \dots, e_7) is a basis of the q -orthogonal complement \mathfrak{m} of $\mathfrak{u}(1)_{k,l}$. The isotropy action of $\mathfrak{u}(1)_{k,l}$ splits \mathfrak{m} into the following irreducible submodules.

$$(18) \quad \begin{aligned} V_1 &:= \text{span}(e_1, e_2) & V_2 &:= \text{span}(e_3, e_4) \\ V_3 &:= \text{span}(e_5, e_6) & V_4 &:= \text{span}(e_7) \end{aligned}$$

The weights of the first three submodules are

$$(19) \quad k - l, 2k + l, k + 2l.$$

If $N^{k,l}$ is generic, V_1 , V_2 , and V_3 are pairwise inequivalent. If $(k, l) = (1, 0)$, V_1 and V_3 are equivalent and V_2 is not equivalent to the two other modules. In the case where $k = l = 1$, V_1 is trivial and V_2 and V_3 are equivalent to each other. Any $SU(3)$ -invariant metric g on $N^{k,l}$ can be identified via q with a $\mathfrak{u}(1)_{k,l}$ -equivariant endomorphism of \mathfrak{m} . We can therefore classify the invariant metrics with the help of Schur's lemma. If $N^{k,l}$ is generic, the matrix representation $g_{ij} := g(e_i, e_j)$ of g is

$$(20) \quad \left(\begin{array}{cc|cc|cc|c} \boxed{a^2} & \boxed{0} & & & & & \\ \boxed{0} & \boxed{a^2} & & & & & \\ & & \boxed{b^2} & \boxed{0} & & & \\ & & \boxed{0} & \boxed{b^2} & & & \\ & & & & \boxed{c^2} & \boxed{0} & \\ & & & & \boxed{0} & \boxed{c^2} & \\ & & & & & & \boxed{f^2} \end{array} \right) \quad \text{with } a, b, c, f \in \mathbb{R} \setminus \{0\}.$$

The coefficients of the $\text{Spin}(7)$ -structure which we will construct contain odd powers of a , b , c , and f . Therefore, we allow these numbers to be negative, too, although this does not change the metric. Next, we assume that $k = 1$ and $l = 0$. The matrix representation of g with respect to the basis $(e_1, e_2, e_5, e_6, e_3, e_4, e_7)$ is

$$(21) \quad \left(\begin{array}{cc|cc|cc|cc|c} \boxed{a^2} & \boxed{0} & \boxed{\beta_{1,5}} & \boxed{\beta_{1,6}} & & & & & \\ \boxed{0} & \boxed{a^2} & \boxed{-\beta_{1,6}} & \boxed{\beta_{1,5}} & & & & & \\ \boxed{\beta_{1,5}} & \boxed{-\beta_{1,6}} & \boxed{c^2} & \boxed{0} & & & & & \\ \boxed{\beta_{1,6}} & \boxed{\beta_{1,5}} & \boxed{0} & \boxed{c^2} & & & & & \\ & & & & \boxed{b^2} & \boxed{0} & & & \\ & & & & \boxed{0} & \boxed{b^2} & & & \\ & & & & & & & & \boxed{f^2} \end{array} \right)$$

with $a, b, c, f, \beta_{1,5}, \beta_{1,6} \in \mathbb{R}$, $a^2 c^2 \geq \beta_{1,5}^2 + \beta_{1,6}^2$, $b \neq 0$, and $f \neq 0$. If $k = l = 1$, the matrix representation of g with respect to $(e_1, e_2, e_7, e_3, e_4, e_5, e_6)$ is

$$(22) \quad \left(\begin{array}{c|cc} \begin{array}{ccc} a_1^2 & \beta_{1,2} & \beta_{1,7} \\ \beta_{1,2} & a_2^2 & \beta_{2,7} \\ \beta_{1,7} & \beta_{2,7} & f^2 \end{array} & & \\ \hline & \begin{array}{cc} b^2 & 0 \\ 0 & b^2 \end{array} & \begin{array}{cc} \beta_{3,5} & \beta_{3,6} \\ -\beta_{3,6} & \beta_{3,5} \end{array} \\ \hline & \begin{array}{cc} \beta_{3,5} & -\beta_{3,6} \\ \beta_{3,6} & \beta_{3,5} \end{array} & \begin{array}{cc} c^2 & 0 \\ 0 & c^2 \end{array} \end{array} \right)$$

As in the other cases, the above matrix has to be positive definite. Throughout the article we assume that the metric on the principal orbit is diagonal with respect to (e_1, \dots, e_7) . This assumption simplifies our calculations and we nevertheless obtain interesting results. Moreover, some of the non-diagonal metrics can be changed by the action of the normalizer $\text{Norm}_{SU(3)}U(1)_{k,l}$ into diagonal ones.

In Eschenburg, Wang [15], Grove, Ziller [19], and Schwachhöfer, Tuschmann [31] it is explained how the Einstein condition $\text{Ric} = \lambda g$ for a cohomogeneity-one manifold can be rewritten as a system of ordinary differential equations for the coefficient functions of g . The Ricci-tensor of a generic $N^{k,l}$ with an arbitrary $SU(3)$ -invariant metric was calculated by Wang [32]. If we put the results of the above papers together, we see that the Einstein condition for a generic $N^{k,l}$ is equivalent to

$$(23) \quad \begin{aligned} & -\frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'}{a} \left(2\frac{a'}{a} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{6}{a^2} - \frac{1}{2} \frac{(k+l)^2}{(k^2+lk+l^2)^2} \frac{f^2}{a^4} \\ & \quad + \frac{a^4-b^4-c^4}{a^2b^2c^2} = \lambda \\ & -\frac{b''}{b} + \frac{b'^2}{b^2} - \frac{b'}{b} \left(2\frac{a'}{a} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{6}{b^2} - \frac{1}{2} \frac{l^2}{(k^2+lk+l^2)^2} \frac{f^2}{b^4} \\ & \quad + \frac{b^4-a^4-c^4}{a^2b^2c^2} = \lambda \\ & -\frac{c''}{c} + \frac{c'^2}{c^2} - \frac{c'}{c} \left(2\frac{a'}{a} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{6}{c^2} - \frac{1}{2} \frac{k^2}{(k^2+lk+l^2)^2} \frac{f^2}{c^4} \\ & \quad + \frac{c^4-a^4-b^4}{a^2b^2c^2} = \lambda \\ & -\frac{f''}{f} + \frac{f'^2}{f^2} - \frac{f'}{f} \left(2\frac{a'}{a} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{1}{2} \frac{(k+l)^2}{(k^2+lk+l^2)^2} \frac{f^2}{a^4} \\ & \quad + \frac{1}{2} \frac{l^2}{(k^2+lk+l^2)^2} \frac{f^2}{b^4} + \frac{1}{2} \frac{k^2}{(k^2+lk+l^2)^2} \frac{f^2}{c^4} = \lambda \\ & -2\frac{a''}{a} - 2\frac{b''}{b} - 2\frac{c''}{c} - \frac{f''}{f} = \lambda \end{aligned}$$

If the principal orbit is $N^{1,0}$ and carries a diagonal metric, the Einstein condition is equivalent to (23) with $k = 1$ and $l = 0$. In particular, a diagonal metric cannot be changed by the equations (23) into a non-diagonal

one for a different value of t . If $k = l = 1$, we do not necessarily have $g(e_1, e_1) = g(e_2, e_2)$ and $\text{Ric} = \lambda g$ becomes

$$\begin{aligned}
(24) \quad & -\frac{a_1''}{a_1} + \frac{a_1'^2}{a_1^2} - \frac{a_1'}{a_1} \left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{6}{a_1^2} - \frac{2}{9} \frac{f^2}{a_1^2 a_2^2} \\
& + 18 \frac{a_1^4 - a_2^4}{a_1^2 a_2^2 f^2} + \frac{a_1^4 - b^4 - c^4}{a_1^2 b^2 c^2} = \lambda \\
& -\frac{a_2''}{a_2} + \frac{a_2'^2}{a_2^2} - \frac{a_2'}{a_2} \left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{6}{a_2^2} - \frac{2}{9} \frac{f^2}{a_1^2 a_2^2} \\
& + 18 \frac{a_2^4 - a_1^4}{a_1^2 a_2^2 f^2} + \frac{a_2^4 - b^4 - c^4}{a_2^2 b^2 c^2} = \lambda \\
& -\frac{b''}{b} + \frac{b'^2}{b^2} - \frac{b'}{b} \left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{6}{b^2} - \frac{1}{18} \frac{f^2}{b^4} \\
& + \frac{b^4 - a_1^4 - c^4}{2a_1^2 b^2 c^2} + \frac{b^4 - a_2^4 - c^4}{2a_2^2 b^2 c^2} = \lambda \\
& -\frac{c''}{c} + \frac{c'^2}{c^2} - \frac{c'}{c} \left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{6}{c^2} - \frac{1}{18} \frac{f^2}{c^4} \\
& + \frac{c^4 - a_1^4 - b^4}{2a_1^2 b^2 c^2} + \frac{c^4 - a_2^4 - b^4}{2a_2^2 b^2 c^2} = \lambda \\
& -\frac{f''}{f} + \frac{f'^2}{f^2} - \frac{f'}{f} \left(\frac{a_1'}{a_1} + \frac{a_2'}{a_2} + 2\frac{b'}{b} + 2\frac{c'}{c} + \frac{f'}{f} \right) + \frac{36}{f^2} - 18 \frac{a_1^2}{a_2^2 f^2} - 18 \frac{a_2^2}{a_1^2 f^2} \\
& + \frac{2}{9} \frac{f^2}{a_1^2 a_2^2} + \frac{1}{18} \frac{f^2}{b^4} + \frac{1}{18} \frac{f^2}{c^4} = \lambda \\
& -\frac{a_1''}{a_1} - \frac{a_2''}{a_2} - 2\frac{b''}{b} - 2\frac{c''}{c} - \frac{f''}{f} = \lambda
\end{aligned}$$

As in the previous case, the differential equations (24) preserve the space of all diagonal metrics on $N^{k,l}$. According to Remark 3.13.5, any formal power series which solves one of the above systems and satisfies the smoothness conditions converges.

Our next step is to deduce a system of ordinary differential equations which is equivalent to $d\Omega = 0$. Let M be a cohomogeneity-one manifold whose principal orbit is a generic Aloff-Wallach space. The following basis of the tangent space induces an $SU(3)$ -invariant $\text{Spin}(7)$ -structure on M whose associated metric restricted to a principal orbit is (20).

$$\begin{aligned}
(25) \quad & f_0 := \frac{\partial}{\partial t} \quad f_1 := \frac{1}{f} e_7 \quad f_2 := \frac{1}{a} e_1 \quad f_3 := \frac{1}{a} e_2 \\
& f_4 := \frac{1}{b} e_4 \quad f_5 := \frac{1}{b} e_3 \quad f_6 := \frac{1}{c} e_6 \quad f_7 := \frac{1}{c} e_5
\end{aligned}$$

The matrix representation of the isotropy action of $\mathfrak{u}(1)_{k,l}$ with respect to the basis (25) can be identified with a one-dimensional subgroup of $\text{Spin}(7)$.

The $SU(3)$ -invariant four-form Ω which is determined by (25) thus is well-defined. Since $(f_i)_{1 \leq i \leq 7}$ is orthonormal with respect to (20), $(f_i)_{1 \leq i \leq 7}$ defines a G_2 -structure on $N^{k,l}$ whose associated metric is an arbitrary $SU(3)$ -invariant one. If we wrote down that G_2 -structure explicitly, we would see that it contains odd powers of a , b , c , and f , as we have remarked above. We calculate $d\Omega$ and see that $d\Omega = 0$ is equivalent to

$$(26) \quad \begin{aligned} \frac{a'}{a} &= \frac{b^2 + c^2 - a^2}{abc} + \frac{-k-l}{2\Delta} \frac{f}{a^2} \\ \frac{b'}{b} &= \frac{c^2 + a^2 - b^2}{abc} + \frac{l}{2\Delta} \frac{f}{b^2} \\ \frac{c'}{c} &= \frac{a^2 + b^2 - c^2}{abc} + \frac{k}{2\Delta} \frac{f}{c^2} \\ \frac{f'}{f} &= -\frac{-k-l}{2\Delta} \frac{f}{a^2} - \frac{l}{2\Delta} \frac{f}{b^2} - \frac{k}{2\Delta} \frac{f}{c^2} \end{aligned}$$

In the above system, Δ denotes $k^2 + lk + l^2$. Furthermore, we have replaced t by $-t$ for cosmetic reasons. This convention will be maintained throughout the article. We remark that the system (26) has also been deduced by Kanno and Yasui [23]. If $k = 1$ and $l = 0$, we can define Ω by (25), too, and $d\Omega = 0$ again is equivalent to (26). In that situation, we can choose the metric on the principal orbit as an arbitrary diagonal one. If $k = l = 1$, f_2 and f_3 have different coefficients, since we may have $g(e_1, e_1) \neq g(e_2, e_2)$. In that case we choose the following basis $(f_i)_{0 \leq i \leq 7}$ which yields an $SU(3)$ -invariant $\text{Spin}(7)$ -structure whose associated metric is an arbitrary diagonal one.

$$(27) \quad \begin{aligned} f_0 &:= \frac{\partial}{\partial t} & f_1 &:= \frac{1}{f} e_7 & f_2 &:= \frac{1}{a_1} e_1 & f_3 &:= \frac{1}{a_2} e_2 \\ f_4 &:= \frac{1}{b} e_4 & f_5 &:= \frac{1}{b} e_3 & f_6 &:= \frac{1}{c} e_6 & f_7 &:= \frac{1}{c} e_5 \end{aligned}$$

The equation $d\Omega = 0$ is equivalent to the slightly more complicated system

$$(28) \quad \begin{aligned} \frac{a'_1}{a_1} &= \frac{b^2 + c^2 - a_1^2}{a_1 bc} + 3 \frac{a_1^2 - a_2^2}{a_1 a_2 f} - \frac{1}{3} \frac{f}{a_1 a_2} \\ \frac{a'_2}{a_2} &= \frac{b^2 + c^2 - a_2^2}{a_2 bc} + 3 \frac{a_2^2 - a_1^2}{a_1 a_2 f} - \frac{1}{3} \frac{f}{a_1 a_2} \\ \frac{b'}{b} &= \frac{1}{2} \frac{a_1^2 + c^2 - b^2}{a_1 bc} + \frac{1}{2} \frac{a_2^2 + c^2 - b^2}{a_2 bc} + \frac{1}{6} \frac{f}{b^2} \\ \frac{c'}{c} &= \frac{1}{2} \frac{a_1^2 + b^2 - c^2}{a_1 bc} + \frac{1}{2} \frac{a_2^2 + b^2 - c^2}{a_2 bc} + \frac{1}{6} \frac{f}{c^2} \\ \frac{f'}{f} &= -3 \frac{(a_1 - a_2)^2}{a_1 a_2 f} + \frac{1}{3} \frac{f}{a_1 a_2} - \frac{1}{6} \frac{f}{b^2} - \frac{1}{6} \frac{f}{c^2} \end{aligned}$$

In the above equations, we again have replaced t by $-t$. The system (28) was also deduced by Kanno and Yasui [24]. We want to know if there are any $SU(3)$ -invariant G_2 -structures on the Aloff-Wallach spaces except those which are determined by a basis of type (25) or (27). Let g be an arbitrary homogeneous metric on $N^{k,l}$. In Lemma 3.8, we have proven that the space of all $SU(3)$ -invariant G_2 -structures on $N^{k,l}$ whose associated metric is g and whose orientation is fixed can be described by

$$(29) \quad \text{Norm}_{SO(7)}U(1)_{k,l}/\text{Norm}_{G_2}U(1)_{k,l} .$$

In this situation, $U(1)_{k,l}$ is identified with its representation on \mathfrak{m} or $\text{Im}(\mathbb{O})$ respectively. We first investigate the problem on the Lie algebra level. Let $G \in \{G_2, SO(7)\}$ and \mathfrak{g} be the Lie algebra of G . The tangent space of $\text{Norm}_G U(1)_{k,l}$ is

$$(30) \quad \text{Norm}_{\mathfrak{g}}\mathfrak{u}(1)_{k,l} := \{x \in \mathfrak{g} \mid [z, x] \in \mathfrak{u}(1)_{k,l} \quad \forall z \in \mathfrak{u}(1)_{k,l}\} .$$

The Lie algebra $\mathfrak{u}(1)_{k,l}$ is generated by a single $z \in \mathfrak{gl}(7, \mathbb{R})$. Let κ be the Killing form of \mathfrak{g} . x is contained in $\text{Norm}_{\mathfrak{g}}\mathfrak{u}(1)_{k,l}$ if and only if

$$(31) \quad [z, x] = \lambda z \quad \text{for a } \lambda \in \mathbb{R} .$$

From this relation, it follows that

$$(32) \quad 0 = \kappa(x, [z, z]) = \kappa([x, z], z) = -\lambda \kappa(z, z) .$$

The above equation is satisfied only if $\lambda = 0$ and thus we have

$$(33) \quad \text{Norm}_{\mathfrak{g}}\mathfrak{u}(1)_{k,l} = \{x \in \mathfrak{g} \mid [z, x] = 0\} =: C_{\mathfrak{g}} \mathfrak{u}(1)_{k,l} .$$

We are going to determine the centralizer $C_{\mathfrak{g}} \mathfrak{u}(1)_{k,l}$. First, we work with the complexification of $C_{\mathfrak{g}} \mathfrak{u}(1)_{k,l}$, since this will simplify some of our arguments. Any $x \in \mathfrak{g} \otimes \mathbb{C}$ has a Cartan decomposition

$$(34) \quad x = h + \sum_{\alpha \in \Phi} \mu_{\alpha} x_{\alpha} \quad \text{with } \mu_{\alpha} \in \mathbb{C} .$$

In the above formula, h is an element of a fixed Cartan subalgebra \mathfrak{h} of $\mathfrak{g} \otimes \mathbb{C}$. Furthermore, Φ is the root system of $\mathfrak{g} \otimes \mathbb{C}$ and x_{α} is a suitable generator of the root space L_{α} of α . We assume without loss of generality that $z \in \mathfrak{h}$. With this notation, the centralizer can be described as follows.

$$(35) \quad C_{\mathfrak{g} \otimes \mathbb{C}}(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C}) = \left\{ x \in \mathfrak{g} \otimes \mathbb{C} \left| [z, x] = \sum_{\alpha \in \Phi} \alpha(z) \mu_{\alpha} x_{\alpha} = 0 \right. \right\}.$$

Let $\Phi' := \{\alpha \in \Phi \mid \alpha(z) = 0\}$. The above formula can be simplified to

$$(36) \quad C_{\mathfrak{g} \otimes \mathbb{C}}(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi'} L_{\alpha}.$$

We specialize to the case $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$. Let $(\theta_1, \theta_2, \theta_3)$ be a basis of the dual \mathfrak{h}^* of \mathfrak{h} such that

$$(37) \quad \Phi = \{\pm\theta_i \mid 1 \leq i \leq 3\} \cup \{\pm\theta_i \pm \theta_j \mid 1 \leq i < j \leq 3\}.$$

For reasons of simplicity we identify \mathfrak{h} and \mathfrak{h}^* by the Killing form. The action of z on \mathfrak{m} is described by the weights (19). For a suitable choice of z and $(\theta_1, \theta_2, \theta_3)$ we thus have

$$(38) \quad z = (k-l)\theta_1 + (2k+l)\theta_2 + (k+2l)\theta_3.$$

Let $\alpha = \sum_{i=1}^3 \alpha_i \theta_i \in \Phi$. The equation $\alpha(z) = 0$ is equivalent to

$$(39) \quad (k-l)\alpha_1 + (2k+l)\alpha_2 + (k+2l)\alpha_3 = 0.$$

$\theta_i(z)$ vanishes if and only if the i^{th} of the three coefficients $k-l, 2k+l, k+2l$ equals zero. Analogously, the root $\pm\theta_i \pm \theta_j$ is contained in Φ' if and only if the i^{th} and the j^{th} coefficient coincide up to the sign. We are now able to describe the normalizer in each of the three cases.

- Let $N^{k,l}$ be a generic Aloff-Wallach space. Since the set Φ' is empty, $\text{Norm}_{\mathfrak{so}(7, \mathbb{C})}(\mathfrak{u}(1)_{k,l} \otimes \mathbb{C}) = \mathfrak{h}$. The normalizer $\text{Norm}_{\mathfrak{so}(7)} \mathfrak{u}(1)_{k,l}$ therefore has to be isomorphic to $3\mathfrak{u}(1)$. More precisely, it has the following matrix representation with respect to the basis $(e'_i)_{1 \leq i \leq 7} := (\frac{e_i}{\|e_i\|})_{1 \leq i \leq 7}$ of \mathfrak{m} .

$$(40) \quad \left\{ \left(\begin{array}{cc|cc|cc|c} 0 & a & & & & & \\ -a & 0 & & & & & \\ \hline & & 0 & b & & & \\ & & -b & 0 & & & \\ \hline & & & & 0 & c & \\ & & & & -c & 0 & \\ \hline & & & & & & 0 \end{array} \right) \mid a, b, c \in \mathbb{R} \right\}.$$

- If $k = 1$ and $l = 0$, the set Φ' consists of the two roots $\pm(\theta_1 - \theta_3)$ and thus is a root system of type A_1 . Since $\text{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{1,0}$ is the compact real form of $\text{Norm}_{\mathfrak{so}(7,\mathbb{C})}(\mathfrak{u}(1)_{1,0} \otimes \mathbb{C})$, it is isomorphic to $\mathfrak{su}(2) \oplus 2\mathfrak{u}(1)$. The semisimple part of the normalizer acts irreducibly on $V_1 \oplus V_3$ such that (e'_1, e'_2, e'_5, e'_6) is identified with the standard basis of \mathbb{C}^2 .
- If $k = l = 1$, Φ' consists of the two pairs $\pm\theta_1$ and $\pm(\theta_2 - \theta_3)$. Since $A_1 \times A_1$ is the Dynkin diagram of $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, the real Lie algebra $\text{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{1,1}$ has to be isomorphic to $2\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. One of the two simple summands acts by its three-dimensional representation on $\text{span}(e'_1, e'_2, e'_7)$. The other one acts irreducibly on $V_2 \oplus V_3$ such that we can identify (e'_3, e'_4, e'_5, e'_6) with the standard basis of \mathbb{C}^2 .

By intersecting $\text{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{k,l}$ with \mathfrak{g}_2 we are able to determine $\text{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{k,l}$. Again, we consider the three cases separately.

- Let $N^{k,l}$ be a generic Aloff-Wallach space. $\text{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{k,l}$ is isomorphic to the two-dimensional subalgebra of (40) which is a Cartan subalgebra of \mathfrak{g}_2 .
- Let $k = 1$ and $l = 0$. By an explicit calculation, we see that the Lie algebra action of the semisimple part of $\text{Norm}_{\mathfrak{so}(7)}\mathfrak{u}(1)_{1,0}$ on $\omega \in \bigwedge^3 \text{Im}(\mathbb{O})^*$ is trivial. The semisimple part therefore is a subalgebra of \mathfrak{g}_2 . Since \mathfrak{g}_2 is of rank two, $\text{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,0}$ is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$.
- Finally, let $k = l = 1$. Since $\text{Norm}_{\mathfrak{g}_2}\mathfrak{u}(1)_{1,1}$ contains the Cartan subalgebra of \mathfrak{g}_2 , it is of rank two. There are four subalgebras of $2\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ which have rank two. One of them is $2\mathfrak{su}(2)$. The other three are isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ where the semisimple part is either an ideal of $2\mathfrak{su}(2)$ or diagonally embedded. By similar techniques as in the previous case, we see that the semisimple part is diagonally embedded. More precisely, it acts irreducibly on $\text{span}(e'_1, e'_2, e'_7)$ and $\text{span}(e'_3, e'_4, e'_5, e'_6)$.

We are now able to describe both normalizers of the Lie group $U(1)_{k,l}$. For the following considerations, we denote the identity component of a Lie group G by G_e . $(\text{Norm}_{SO(7)}U(1)_{k,l})_e$ acts transitively and almost freely on any connected component of $\text{Norm}_{SO(7)}U(1)_{k,l}/\text{Norm}_{G_2}U(1)_{k,l}$. For our purpose, it therefore suffices to describe the quotient $(\text{Norm}_{SO(7)}U(1)_{k,l})_e/(\text{Norm}_{G_2}U(1)_{k,l})_e$, which covers any connected component of $\text{Norm}_{SO(7)}U(1)_{k,l}/\text{Norm}_{G_2}U(1)_{k,l}$.

- If $N^{k,l}$ is generic, we have

$$(41) \quad (\text{Norm}_{SO(7)}U(1)_{k,l})_e/(\text{Norm}_{G_2}U(1)_{k,l})_e \cong U(1)^3/U(1)^2 \cong S^1.$$

Let T be a one-dimensional connected Lie subgroup of the maximal torus of $SO(7)$ whose Lie algebra is (40). We choose T in such a way that $G_2 \cap T$ is discrete. The action of T on a fixed homogeneous G_2 -structure ω generates a connected component of the space of all $SU(3)$ -invariant G_2 -structures which have the same associated metric and orientation as ω .

- If $k = 1$ and $l = 0$, we have

$$(42) \quad \begin{aligned} (\text{Norm}_{SO(7)} U(1)_{k,l})_e / (\text{Norm}_{G_2} U(1)_{k,l})_e &\cong (U(2) \times U(1)) / U(2) \\ &\cong S^1. \end{aligned}$$

The space of all $SU(3)$ -invariant G_2 -structures with a fixed associated metric and orientation can therefore be described as in the previous case.

- The semisimple part of $\text{Norm}_{\mathfrak{g}_2} \mathfrak{u}(1)_{1,1}$ is diagonally embedded into the semisimple part $2\mathfrak{su}(2)$ of $\text{Norm}_{\mathfrak{so}(7)} \mathfrak{u}(1)_{1,1}$. We denote the ideal of $2\mathfrak{su}(2)$ which acts non-trivially on $\text{span}(e'_1, e'_2, e'_7)$ by $\mathfrak{su}(2)'$. The abelian part of $\text{Norm}_{\mathfrak{g}_2} \mathfrak{u}(1)_{1,1}$ is not a subgroup of $\mathfrak{su}(2)'$. We can conclude that the Lie group which corresponds to $\mathfrak{su}(2)'$ is isomorphic to $SO(3)$ and acts transitively and freely on $(\text{Norm}_{SO(7)} U(1)_{k,l})_e / (\text{Norm}_{G_2} U(1)_{k,l})_e$. The set of all $SU(3)$ -invariant G_2 -structures on $N^{1,1}$ whose associated metric and orientation is fixed can therefore be generated by an $SO(3)$ -action on a single G_2 -structure.

Until now, we have only determined the type of the connected components of $\text{Norm}_{SO(7)} U(1)_{k,l} / \text{Norm}_{G_2} U(1)_{k,l}$, but not their number. We therefore cannot rule out that there are further homogeneous G_2 -structures which cannot be obtained by the above $U(1)$ - or $SO(3)$ -actions. It may be possible that the additional G_2 -structures could be extended to new examples of parallel cohomogeneity-one $\text{Spin}(7)$ -structures. We will nevertheless not discuss this issue further. There are three reasons for our decision. First, the study of the $\text{Spin}(7)$ -structures whose restriction to a principal orbit is one of the G_2 -structures which we have constructed so far is already a rewarding project even if further examples existed. Second, it may be possible that we switch to another connected component of $\text{Norm}_{SO(7)} U(1)_{k,l} / \text{Norm}_{G_2} U(1)_{k,l}$ if we change the sign of some of the functions a, a_1, a_2, b, c , and f . Therefore, we may already describe more than one or even all connected components by our ansatz for $(f_i)_{1 \leq i \leq 7}$. A third reason is that the results which we have found are sufficient to make statements on the holonomy of our metrics.

We start with the generic Aloff-Wallach spaces $N^{k,l}$. Let g be a fixed metric on $N^{k,l}$ and ω be a G_2 -structure whose associated metric is g . By an explicit calculation we see that ω is cocalibrated only if it is induced by the basis $(f_i)_{1 \leq i \leq 7}$ of \mathfrak{m} which we have defined on page 17. Let \tilde{g} be a cohomogeneity-one metric whose holonomy is contained in $\text{Spin}(7)$. We assume that there is a principal orbit such that the restriction of \tilde{g} to that orbit is g . Since

the holonomy bundle is $SU(3)$ -invariant, any parallel $\text{Spin}(7)$ -structure Ω whose associated metric is \tilde{g} is $SU(3)$ -invariant, too. Any parallel $\text{Spin}(7)$ -structure induces a cocalibrated G_2 -structure on the principal orbit. Since the set of all $SU(3)$ -invariant cocalibrated G_2 -structures is discrete, the set of all invariant parallel $\text{Spin}(7)$ -structures is discrete, too. According to Lemma 3.14, the holonomy of \tilde{g} cannot be $SU(4)$ or one of its subgroups.

If the holonomy was G_2 , there would exist a parallel vector field X on the manifold. Moreover, the space of all parallel vector fields would be one-dimensional. Since that space is $SU(3)$ -invariant, X is invariant, too, and of type $c_1(t) \cdot \frac{\partial}{\partial t} + \frac{c_2(t)}{f(t)} \cdot e_7$. The dual $c_1(t) \cdot dt + c_2(t) \cdot f(t) \cdot e^7$ of X has to be a closed one-form. By calculating the exterior derivative, we see that c_2 has to vanish for all t . Since the length of X has to be constant, c_1 has to be constant, too. If $\frac{\partial}{\partial t}$ was parallel, we would have $a' = b' = c' = f' = 0$. In that situation, the manifold would be the product of $N^{k,l}$ with a parallel $SU(3)$ -invariant G_2 -structure and an interval. This is impossible since $N^{k,l}$ is not a torus and we thus have proven that the holonomy is exactly $\text{Spin}(7)$.

If the principal orbit is $N^{1,0}$, it follows by the same arguments that the holonomy is all of $\text{Spin}(7)$. In the case where $k = l = 1$, the situation is more complicated, since the space of all G_2 -structures with a fixed associated metric is three-dimensional. Let ω be the G_2 -structure on $N^{1,1}$ which is induced by the basis $(f_i)_{1 \leq i \leq 7}$ from page 18. If the holonomy is contained in $SU(4)$, there exists a map

$$(43) \quad \tilde{\omega} : [0, \epsilon) \rightarrow \bigwedge^3 \mathfrak{m}^*$$

such that $\tilde{\omega}(0) = \omega$, $d * \tilde{\omega}(s) = 0$ for all $s \in [0, \epsilon)$, and the metric which is associated to $\tilde{\omega}(s)$ is the same as of ω . We have proven that $\tilde{\omega}(s)$ can be obtained by the action of an $A(s) \in SO(3)$ on (e'_1, e'_2, e'_7) . We consider the case where $A(s)$ is of type

$$(44) \quad \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With the help of a short MAPLE program, we can show that if $s \notin \pi\mathbb{Z}$, $d * \tilde{\omega}(s) = 0$ is equivalent to $a_1(t) + a_2(t) = 0$. Under this assumption, it follows from (28) that $a_1(t)^2 = b(t)^2 + c(t)^2$. Under these assumptions (28) is explicitly solvable and we obtain the metrics of Bazaikin and Malkovich [6]. In [6] it is proven that the holonomy is all of $SU(4)$, except in a limiting case where we obtain Calabis [10] hyperkähler metric on $T^*\mathbb{CP}^2$.

If we replace the A from (44) by another one-parameter subgroup of $SO(3)$, we obtain more complicated conditions on the $\text{Spin}(7)$ -structures, which may

or may not be satisfiable. The question if there are further metrics whose holonomy is a proper subgroup of $\text{Spin}(7)$ is a subject of future research.

At the end of this section, we classify all possible singular orbits of a cohomogeneity-one manifold whose principal orbit is an Aloff-Wallach space. This classification can also be found in Gambioli [18]. Our results are summarized by the following lemma.

Lemma 4.2. *Let $U(1)$ be embedded into $SU(3)$ as $U(1)_{k,l}$. Furthermore, let K be a connected, closed group with $U(1)_{k,l} \subsetneq K \subseteq SU(3)$. The Lie algebra of K we denote by \mathfrak{k} . In this situation, \mathfrak{k} and K can be found in the table below. Moreover, $K/U(1)_{k,l}$ and $SU(3)/K$ satisfy the following topological conditions.*

\mathfrak{k}	K	$K/U(1)_{k,l}$	$SU(3)/K$	Condition on k and l
$2\mathfrak{u}(1)$	$U(1)^2$	$\cong S^1$	$= SU(3)/U(1)^2$	
$\mathfrak{su}(2)$	$SU(2)$	$\cong S^2$	$\cong S^5$	$k \cdot l \cdot (-k - l) = 0$
$\mathfrak{su}(2)$	$SO(3)$	$\cong S^2$	$= SU(3)/SO(3)$	$k \cdot l \cdot (-k - l) = 0$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\cong S^3/\mathbb{Z}_{ k+l }$	$\cong \mathbb{CP}^2$	$(k, l) \neq (1, -1)$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\cong S^2 \times S^1$	$\cong \mathbb{CP}^2$	$(k, l) = (1, -1)$
$\mathfrak{su}(3)$	$SU(3)$	$= N^{k,l} \not\cong S^7/\Gamma$		

In the above table, Γ denotes an arbitrary discrete subgroup of $O(8)$ and the group $\mathbb{Z}_{|k+l|}$, by which we divide S^3 , is explicitly described by (46).

Proof. The fact that \mathfrak{k} has to be an $\mathfrak{u}(1)_{k,l}$ -module, reduces the number of subspaces of $\mathfrak{su}(3)$ which we have to consider. We have to check for all $\mathfrak{u}(1)_{k,l}$ -modules \mathfrak{k} with $\mathfrak{u}(1)_{k,l} \subsetneq \mathfrak{k} \subseteq \mathfrak{su}(3)$ if they are closed under the Lie bracket and if $K/U(1)_{k,l}$ is covered by a sphere. If $N^{k,l}$ is an exceptional Aloff-Wallach space, \mathfrak{m} contains pairs (U, U') of equivalent submodules. In that situation, there are infinitely many submodules of \mathfrak{k} which are transversely embedded into $U \oplus U'$. Therefore, we often have to distinguish between the different types of Aloff-Wallach spaces in the course of this proof. We consider each of the possible values of $\dim \mathfrak{k}$ separately.

$\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, $\dim W = 1$: If $(k, l) \neq (1, 1)$, V_4 is the only one-dimensional submodule of \mathfrak{m} and the statements of the lemma are satisfied. If $k = l = 1$, V_1 is trivial. We can choose the generator of W as an arbitrary $x \in (V_1 \oplus V_4) \setminus \{0\}$ and obtain the same statements on the singular orbit.

$\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, $\dim W = 2$: If $N^{k,l}$ is generic, the only two-dimensional submodules of \mathfrak{m} are V_1 , V_2 , and V_3 . In any case \mathfrak{k} is not closed under the Lie bracket.

Next, we assume that $k = 1$ and $l = 0$. $\mathfrak{u}(1)_{1,0} \oplus V_2$ is a Lie algebra, which is isomorphic to $\mathfrak{su}(2)$ and we have $K = SU(2)$ and $K/U(1)_{1,0} \cong S^2$. Since the modules V_1 and V_3 are $\mathfrak{u}(1)_{1,0}$ -equivariantly isomorphic, there may exist a two-dimensional space W which is transversely embedded into $V_1 \oplus V_3$ such

that \mathfrak{k} is closed under the Lie bracket. It turns out that the possible \mathfrak{k} are precisely

$$(45) \quad \mathfrak{k} = \text{span}(e_8, \frac{1}{2}\sqrt{2}e_1 + \frac{1}{2}\sqrt{2}e_5, -\frac{1}{2}\sqrt{2}e_2 - \frac{1}{2}\sqrt{2}e_6)$$

and its conjugates with respect to $U(1)_{1,1}$. Since \mathfrak{k} is conjugate to the standard embedding of $\mathfrak{so}(3)$ into $\mathfrak{su}(3)$, we have $K \cong SO(3)$, $K/U(1)_{1,-1} \cong S^2$ and the singular orbit is the symmetric space $SU(3)/SO(3)$.

If $k = l = 1$, \mathfrak{k} again cannot be a Lie algebra.

$\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, $\dim W = 3$: Let $N^{k,l}$ be a generic Aloff-Wallach space. In this situation, we have $W = V_i \oplus V_4$ for an $i \in \{1, 2, 3\}$. As a Lie algebra \mathfrak{k} is isomorphic to $\mathfrak{u}(2)$. We assume without loss of generality that $i = 1$. The Lie group K is given by $S(U(2) \times U(1))$.

We analyze the topology of $K/U(1)_{k,l}$. Let $\pi : SU(2) \rightarrow K/U(1)_{k,l}$ be the map with $\pi(h) := hU(1)_{k,l}$, where $SU(2)$ is embedded into $SU(3)$ such that its Lie algebra is $\mathfrak{u}(1)_{1,-1} \oplus V_1$. It is easy to see that π is a covering map. Its kernel is $SU(2) \cap U(1)_{k,l}$. Since k and l are coprime, this intersection is, except for $(k, l) = (1, -1)$,

$$(46) \quad \left\{ \left(\begin{array}{cc} e^{2\pi i \frac{m}{k+l}} & 0 \\ 0 & e^{-2\pi i \frac{m}{k+l}} \end{array} \right) \mid m \in \mathbb{Z} \right\}.$$

The quotient $K/U(1)_{k,l}$ thus is the lens space $L(k+l, 1)$. For reasons of brevity, we will denote $K/U(1)_{k,l}$ by $S^3/\mathbb{Z}_{|k+l|}$, since $\mathbb{Z}_{|k+l|}$ will always be the above discrete group. The quotient $SU(3)/K$ is diffeomorphic to \mathbb{CP}^2 .

Next, we consider the exceptional case $k = 1$, $l = 0$. If we choose W as $V_i \oplus V_4$ with $i \in \{1, 3\}$, $K/U(1)_{1,0}$ is the sphere S^3 . For $W = V_2 \oplus V_4$, we have $K/U(1)_{1,0} = S^2 \times S^1$. In order to finish the first exceptional case, we have to check if we can choose $W = W' \oplus V_4$ with $W' \subseteq V_1 \oplus V_3$ but $W' \notin \{V_1, V_3\}$. Since the Cartan subalgebra $\text{span}(e_7, e_8)$, which we shortly denote by $2\mathfrak{u}(1)$, is contained in \mathfrak{k} , W' has to be a $2\mathfrak{u}(1)$ -module. The q -orthogonal complement of $2\mathfrak{u}(1)$ in $\mathfrak{su}(3)$ decomposes with respect to $2\mathfrak{u}(1)$ into $V_1 \oplus V_2 \oplus V_3$ and the three modules are pairwise inequivalent. Therefore, the case $W' \notin \{V_1, V_3\}$ cannot happen.

We finally consider the case $k = l = 1$. \mathfrak{k} has to be isomorphic to the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$. As above, we decompose \mathfrak{k} into the direct sum $\mathfrak{h} \oplus W'$ of $\mathfrak{u}(1)_{1,1}$ -modules, where \mathfrak{h} is a Cartan subalgebra of \mathfrak{k} which contains $\mathfrak{u}(1)_{1,1}$ and W' is its q -orthogonal complement. Since $\text{span}(e_1, e_2, e_7)$ is the centralizer of $\mathfrak{u}(1)_{1,1}$, \mathfrak{h} can be any of the following spaces.

$$(47) \quad \mathfrak{h} = \mathfrak{u}(1)_{1,1} \oplus \text{span}(x) \quad \text{with } x \in \text{span}(e_1, e_2, e_7) \setminus \{0\}$$

W' is either a submodule of $\text{span}(e_1, e_2, e_7)$ or of $V_2 \oplus V_3$. In the first case, it has to be the complement of $\text{span}(x)$ in $\text{span}(e_1, e_2, e_7)$ and \mathfrak{k} therefore is $2\mathfrak{u}(1) \oplus V_1$, which yields $K/U(1)_{1,1} = S^3/\mathbb{Z}_2$ and $SU(3)/K = \mathbb{CP}^2$.

We turn to the case where $W' \subseteq V_2 \oplus V_3$. There exists an

$$(48) \quad A \in \left\{ \left(\begin{array}{c|c} A' & \\ \hline & 1 \end{array} \right) \mid A' \in SU(2) \right\} \quad \text{with } A \mathfrak{h} A^{-1} = 2\mathfrak{u}(1).$$

We conjugate our decomposition of \mathfrak{k} by A and obtain

$$(49) \quad A \mathfrak{k} A^{-1} = 2\mathfrak{u}(1) \oplus AW' A^{-1}.$$

Since $AW' A^{-1}$ is a $2\mathfrak{u}(1)$ -module, we can prove by the same arguments as in the case $(k, l) = (1, 0)$ that $AW' A^{-1}$ is either V_2 or V_3 . In both cases we obtain $K/U(1)_{1,1} = S^3$ and $SU(3)/K = \mathbb{CP}^2$.

$\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, $\dim W \in \{4, 6\}$: Since for any root α of \mathfrak{k} , $-\alpha$ is a root, too, we have $\dim \mathfrak{k} \equiv \text{rank } \mathfrak{k} \pmod{2}$. Therefore, the rank of \mathfrak{k} has to be odd. Since $\text{rank } \mathfrak{su}(3) = 2$, the only possibility for $\text{rank } \mathfrak{k}$ is in fact 1. Since the only Lie algebras of rank 1 which belong to a compact Lie group are $\mathfrak{u}(1)$ and $\mathfrak{su}(2)$, we can exclude these cases.

$\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, $\dim W = 5$: The only six-dimensional Lie algebra of rank ≤ 2 which belongs to a compact Lie group is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. There is no Lie subalgebra of $\mathfrak{su}(3)$ of this type and we therefore can exclude this case.

$\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus W$, $\dim W = 7$: In this case, \mathfrak{k} is all of $\mathfrak{su}(3)$. Since $H^4(N^{k,l}, \mathbb{Z}) = \mathbb{Z}_{k^2+kl+l^2}$, $N^{k,l}$ is not a quotient of S^7 by a discrete subgroup $\Gamma \subseteq O(8)$. \square

Convention 4.3. We want to treat the three Lie algebras $2\mathfrak{u}(1) \oplus V_i$ with $i \in \{1, 2, 3\}$ in a uniform manner. Therefore, we will drop the convention $k \geq l \geq 0$ whenever we consider the case $\mathfrak{k} \cong \mathfrak{u}(2)$. This will allow us to fix \mathfrak{k} as $2\mathfrak{u}(1) \oplus V_1$. Since we can replace (k, l) by $(-k, -l)$, we can still assume that $k \geq l$. We have implicitly used this convention in the statement of the lemma when we described the topology of $K/U(1)_{k,l}$ as $S^3/\mathbb{Z}_{|k+l|}$ instead of $S^3/\mathbb{Z}_{|k|}$ or $S^3/\mathbb{Z}_{|l|}$, which would be the case if $i \in \{2, 3\}$.

The results of this section can be summarized as follows.

Theorem 4.4. *Let (M, Ω) be a $Spin(7)$ -orbifold with a cohomogeneity-one action of $SU(3)$ which preserves Ω and whose principal orbit is an Aloff-Wallach space. The metric associated to Ω we denote by g . In this situation, the following statements are true.*

- (1) (a) If $N^{k,l}$ is generic, the matrix representation of g with respect to the basis (e_1, \dots, e_7) from page 14 is of type (20).
- (b) If $k = 1$ and $l = 0$, the matrix representation of g with respect to the basis $(e_1, e_2, e_5, e_6, e_3, e_4, e_7)$ is of type (21).
- (c) If $k = 1$ and $l = 1$, the matrix representation of g with respect to the basis $(e_1, e_2, e_7, e_3, e_4, e_5, e_6)$ is of type (22).
- (2) We assume that $N^{k,l}$ is generic or $N^{1,0}$ and that g is diagonal with respect to the basis from page 14. Let \mathcal{S}' be a connected component of the space \mathcal{S} of all $SU(3)$ -invariant $Spin(7)$ -structures on M with the same associated metric as Ω . We assume that \mathcal{S}' contains a four-form $\tilde{\Omega}$ which is obtained from a basis of type (25). In this situation, Ω can be parallel only if it coincides with $\tilde{\Omega}$. In other words, (25) is up to other connected components of \mathcal{S} the only possible basis for a diagonal metric with holonomy contained in $Spin(7)$.
- (3) (a) If $N^{k,l}$ is generic or $N^{1,0}$ and Ω is determined by a basis of type (25), Ω is parallel if and only if the system (26) is satisfied. In that case, the holonomy of g is all of $Spin(7)$.
- (b) If $k = l = 1$ and Ω is determined by a basis of type (27), Ω is parallel if and only if the system (28) is satisfied. In this case, the holonomy of g is a subgroup of $Spin(7)$. If we also have $a_1(t) + a_2(t) = 0$ for all t , the holonomy of g is $SU(4)$ except in the case where we obtain the Calabi metric which has holonomy $Sp(2)$.

If M has a singular orbit, which has to be the case if (M, Ω) is parallel and complete, it can be found in the table of Lemma 4.2. For any choice of the principal and the singular orbit, the orbifold M is a manifold, except in the case where the singular orbit is \mathbb{CP}^2 and $|k + l| \neq 1$. In that case, M is a $D^4/\mathbb{Z}_{|k+l|}$ -bundle over \mathbb{CP}^2 .

In the following three sections, we search for cohomogeneity-one metrics with special holonomy on the spaces of the above theorem. We have to treat each possible combination of a principal and a singular orbit as a separate case, since the equations for the holonomy reduction or their initial conditions will change.

5. $SU(3)/U(1)^2$ AS SINGULAR ORBIT

5.1. The generic Aloff-Wallach spaces as principal orbit. Throughout this and the following sections we will assume that the singular orbit is at $t = 0$. In the situation of this subsection, the Lie algebra of the isotropy group at the singular orbit is given by $\mathfrak{u}(1)_{k,l} \oplus V_4$. We therefore have $f(0) = 0$ and $a(0), b(0), c(0) \neq 0$ as initial conditions. We take a look at the equation

$$(50) \quad f' = -\frac{-k-l}{2\Delta} \frac{f^2}{a^2} - \frac{l}{2\Delta} \frac{f^2}{b^2} - \frac{k}{2\Delta} \frac{f^2}{c^2}$$

from the system (26). We see that $f'(0) = 0$ and by a complete induction we can prove that all higher derivatives of f at $t = 0$ vanish, too. Since the metric has to be analytic, it follows that f vanishes identically. If we insert $f \equiv 0$ into (26), we obtain the equations for a cohomogeneity-one metric with principal orbit $SU(3)/U(1)^2$ and holonomy G_2 . Since those equations were investigated by Cleyton and Swann [11], we will not discuss this issue further.

Our next aim is to classify all cohomogeneity-one Einstein metrics with our fixed orbit structure. In order to apply Theorem 3.12, we have to decompose certain $2\mathfrak{u}(1)$ -modules. The fact that we need these decompositions in the later sections anyway is a further motivation for our task. As in Section 3, we denote the tangent space of the singular orbit by \mathfrak{p} and its normal space by \mathfrak{p}^\perp . We need to describe the spaces W_m^h and W_2^v of all $2\mathfrak{u}(1)$ -equivariant maps from $S^m(\mathfrak{p}^\perp)$ and $S^2(\mathfrak{p}^\perp)$ into $S^2(\mathfrak{p})$ and $S^2(\mathfrak{p})^\perp$. Therefore, we decompose $S^2(\mathfrak{p})$ into irreducible $2\mathfrak{u}(1)$ -submodules. \mathfrak{p} is the direct sum of the modules V_1 , V_2 , and V_3 , whose weights are given by

$$(51) \quad V_1 = \mathbb{V}_{3k+3l, k-l}, \quad V_2 = \mathbb{V}_{3l, 2k+l}, \quad V_3 = \mathbb{V}_{-3k, k+2l}.$$

In the above formula $\mathbb{V}_{r,s}$ denotes the two-dimensional real $2\mathfrak{u}(1)$ -module on which e_7 acts with weight r and e_8 acts with weight s . With the help of the relations

$$(52) \quad \begin{aligned} S^2(\mathbb{V}_{r,s}) &= \mathbb{V}_{2r, 2s} \oplus \mathbb{R} \\ \mathbb{V}_{r_1, s_1} \otimes \mathbb{V}_{r_2, s_2} &= \mathbb{V}_{r_1+r_2, s_1+s_2} \oplus \mathbb{V}_{r_1-r_2, s_1-s_2} \\ \mathbb{V}_{r,s} &= \overline{\mathbb{V}_{-r, -s}} \cong \mathbb{V}_{-r, -s} \end{aligned}$$

we see that

$$(53) \quad \begin{aligned} S^2(\mathfrak{p}) &= \mathbb{V}_{6k+6l, 2k-2l} \oplus \mathbb{V}_{6l, 4k+2l} \oplus \mathbb{V}_{-6k, 2k+4l} \\ &\oplus \mathbb{V}_{3k+6l, 3k} \oplus \mathbb{V}_{3k, -k-2l} \oplus \mathbb{V}_{3l, 2k+l} \oplus \mathbb{V}_{6k+3l, -3l} \\ &\oplus \mathbb{V}_{-3k+3l, 3k+3l} \oplus \mathbb{V}_{3k+3l, k-l} \oplus 3\mathbb{R}. \end{aligned}$$

Next, we have to describe the action of $2\mathfrak{u}(1)$ on \mathfrak{p}^\perp . The tangent vector $\frac{\partial}{\partial t} \in \mathfrak{p}^\perp$ is fixed by the action of $U(1)_{k,l}$. e_8 therefore acts trivially on \mathfrak{p}^\perp . $e_7 \in 2\mathfrak{u}(1)$ generates the group $U(1)_{2l+k, -2k-l} \subseteq SU(3)$, which we will shortly denote by $U(1)'$. The circle $U(1)^2/U(1)_{k,l}$, where $U(1)^2$ as usual denotes the subgroup of all diagonal matrices in $SU(3)$, can be considered as a subset of \mathfrak{p}^\perp . Moreover, the orbit of the $U(1)'$ -action on a point $p \in \mathfrak{p}^\perp \setminus \{0\}$

can be identified with a loop in the space $U(1)^2/U(1)_{k,l}$. The weight of the $U(1)'$ -action on \mathfrak{p}^\perp coincides with the homotopy class of that loop, which is an integer. That number is the same as the number of all $t \in [0, 2\pi)$ such that $\exp(t \cdot e_7) \in U(1)_{k,l}$. After a short calculation we can conclude that

$$(54) \quad \mathfrak{p}^\perp = \mathbb{V}_{2(k^2+lk+l^2),0}.$$

We are now able to decompose $S^m(\mathfrak{p}^\perp)$ into irreducible $2\mathfrak{u}(1)$ -submodules and obtain

$$(55) \quad S^m(\mathfrak{p}^\perp) = \begin{cases} \mathbb{V}_{2m(k^2+lk+l^2),0} \oplus \mathbb{V}_{2(m-2)(k^2+lk+l^2),0} \\ \oplus \dots \oplus \mathbb{V}_{4(k^2+lk+l^2),0} \oplus \mathbb{R} & \text{if } m \text{ is even} \\ \mathbb{V}_{2m(k^2+lk+l^2),0} \oplus \mathbb{V}_{2(m-2)(k^2+lk+l^2),0} \\ \oplus \dots \oplus \mathbb{V}_{2(k^2+lk+l^2),0} & \text{if } m \text{ is odd} \end{cases}$$

With the help of the decompositions of $S^2(\mathfrak{p})$ and $S^m(\mathfrak{p}^\perp)$ as well as Schur's lemma it follows that

$$(56) \quad \dim W_m^h = \begin{cases} 3 & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases} \quad \text{and} \quad \dim W_2^v = 3.$$

Since \mathfrak{p}^\perp is trivial with respect to $U(1)_{k,l}$ and $N^{k,l}$ is not exceptional, \mathfrak{p} and \mathfrak{p}^\perp have no non-trivial $U(1)_{k,l}$ -submodule in common and Assumption 3.11 is satisfied. We can therefore apply Theorem 3.12 and Remark 3.13 to our situation and thus have proven the following theorem.

Theorem 5.1. *In the situation of Theorem 4.4, let $N^{k,l}$ be generic and let $SU(3)/U(1)^2$ be a singular orbit at $t = 0$.*

- (1) *There exists no $SU(3)$ -invariant metric on M which has holonomy $Spin(7)$.*
- (2) *For any choice of $a_0, b_0, c_0 \in \mathbb{R} \setminus \{0\}$ and $f_3, \lambda \in \mathbb{R}$, there exists a unique $SU(3)$ -invariant Einstein metric on a tubular parameterize of $SU(3)/U(1)^2$ such that*
 - (a) $a(0)^2 = a_0^2, b(0)^2 = b_0^2, c(0)^2 = c_0^2,$
 - (b) $f'''(0) = f_3,$ and
 - (c) *the Einstein constant is λ .*

5.2. $N^{1,0}$ as principal orbit. In this situation, the only trivial $U(1)_{1,0}$ -submodule of \mathfrak{m} again is V_4 and we have $f(0) = 0$ as initial condition. Since we restrict ourselves to diagonal metrics, we can work as before with the system (26). Therefore, we can prove by the same arguments as in the previous case that f vanishes and the metric is degenerate.

We apply the methods of [15] in order to describe the cohomogeneity-one Einstein metrics with the orbit structure of this subsection. Since none of the $U(1)_{1,0}$ -modules V_1 , V_2 , and V_3 is trivial, Assumption 3.11 is satisfied and we are in the situation of Theorem 3.12. If $k = 1$ and $l = 0$, the decompositions of $S^2(\mathfrak{p})$ and $S^m(\mathfrak{p}^\perp)$ which we have found specialize to

$$(57) \quad \begin{aligned} S^2(\mathfrak{p}) = & \mathbb{V}_{6,2} \oplus \mathbb{V}_{0,4} \oplus \mathbb{V}_{-6,2} \\ & \oplus \mathbb{V}_{3,3} \oplus \mathbb{V}_{3,-1} \oplus \mathbb{V}_{0,2} \oplus \mathbb{V}_{6,0} \\ & \oplus \mathbb{V}_{-3,3} \oplus \mathbb{V}_{3,1} \oplus 3\mathbb{R} \end{aligned}$$

and

$$(58) \quad S^m(\mathfrak{p}^\perp) = \begin{cases} \mathbb{V}_{2m,0} \oplus \mathbb{V}_{2(m-2),0} \oplus \dots \oplus \mathbb{V}_{4,0} \oplus \mathbb{R} & \text{if } m \text{ is even} \\ \mathbb{V}_{2m,0} \oplus \mathbb{V}_{2(m-2),0} \oplus \dots \oplus \mathbb{V}_{2,0} & \text{if } m \text{ is odd} \end{cases}$$

$S^2(\mathfrak{p})$ contains a summand which is isomorphic to $\mathbb{V}_{6,0}$. Therefore, we have $\dim W_m^h = 2$ if m is odd and ≥ 3 . We calculate $\dim W_m^h$ for all $m \in \mathbb{N}_0$ and obtain

$$(59) \quad \dim W_m^h = \begin{cases} 3 & \text{if } m \text{ is even} \\ 0 & \text{if } m = 1 \\ 2 & \text{if } m \geq 3 \text{ is odd} \end{cases}$$

There are two initial conditions of 3^{rd} order which we can choose freely. Since the summand $\mathbb{V}_{6,0}$ is contained in $V_1 \otimes V_3$, the two parameters of 3^{rd} order describe how the metric on the singular orbit, which is diagonal, changes into a non-diagonal one. Analogously to the previous subsection we have proven the following theorem.

Theorem 5.2. *In the situation of Theorem 4.4, let $N^{1,0}$ be the principal orbit and let $SU(3)/U(1)^2$ be a singular orbit at $t = 0$.*

- (1) *There exists no $SU(3)$ -invariant metric on M which is diagonal with respect to the basis from page 14 and has holonomy $Spin(7)$.*
- (2) *For any choice of $a_0, b_0, c_0 \in \mathbb{R} \setminus \{0\}$ and $\beta, \tilde{\beta}, f_3, \lambda \in \mathbb{R}$, there exists a unique $SU(3)$ -invariant Einstein metric on a tubular neighborhood of $SU(3)/U(1)^2$ such that*
 - (a) $a(0)^2 = a_0^2$, $b(0)^2 = b_0^2$, $c(0)^2 = c_0^2$,
 - (b) $\beta_{1,5}'''(0) = \beta$, $\beta_{1,6}'''(0) = \tilde{\beta}$,
 - (c) $f'''(0) = f_3$, and
 - (d) *the Einstein constant is λ .*

5.3. $N^{1,1}$ as principal orbit. We make the same assumptions as in Theorem 4.4 and thus can work with the system (28). The isotropy algebra \mathfrak{k} of the $SU(3)$ -action on the singular orbit is spanned by e_8 and an arbitrary $x \in V_1 \oplus V_4 \setminus \{0\}$. We assume that $x = e_7$ or equivalently that $f(0) = 0$. Since the length of the collapsing circle shall be $2\pi t + O(t^2)$ for small t , we need $f'(0) \neq 0$. The differential equation for f' contains the additional term $-3\frac{(a_1-a_2)^2}{a_1a_2}$, which is not there if the principal orbit is generic. We therefore have $f'(0) \neq 0$ if and only if $a_1(0) = a_0 = -a_2(0)$ for an $a_0 \in \mathbb{R} \setminus \{0\}$. The values b_0 of $b(0)$ and c_0 of $c(0)$ can be chosen arbitrarily. We thus have formulated an initial value problem for the first order system (28).

We take a short look at the case where $x = \alpha e_1 + \beta e_2 + \gamma e_7$ for some coefficients $\alpha, \beta, \gamma \in \mathbb{R}$. Since we want $\lim_{t \rightarrow 0} g_t(x, x) = 0$, it follows that $x \in \{e_1, e_2, e_7\}$. There exists a $\tau \in \text{Norm}_{SU(3)}U(1)_{1,1}$ which maps e_1 or e_2 to e_7 . Metrics which are described by solutions of (28) with $a_1(0) = 0$ or $a_2(0) = 0$ can be therefore be mapped by τ to metrics with $f(0) = 0$. Diagonal metrics with $a_1(0) = 0$ or $a_2(0) = 0$ can be mapped to non-diagonal metrics, which we will not consider in the article. Nevertheless, the above observation is a motivation to restrict ourselves to the case $f(0) = 0$.

We make a power series ansatz for (28) with the initial values a_0, b_0 , and c_0 and obtain

$$\begin{aligned}
 (60) \quad a_1(t) &= a_0 - \frac{1}{2} \frac{a_0^2 - b_0^2 - c_0^2}{b_0 c_0} t + \frac{1}{8} \frac{3a_0^4 - 2a_0^2 b_0^2 - 2a_0^2 c_0^2 - b_0^4 + 14b_0^2 c_0^2 - c_0^4}{a_0 b_0^2 c_0^2} t^2 + \dots \\
 a_2(t) &= -a_0 - \frac{1}{2} \frac{a_0^2 - b_0^2 - c_0^2}{b_0 c_0} t - \frac{1}{8} \frac{3a_0^4 - 2a_0^2 b_0^2 - 2a_0^2 c_0^2 - b_0^4 + 14b_0^2 c_0^2 - c_0^4}{a_0 b_0^2 c_0^2} t^2 + \dots \\
 b(t) &= b_0 + 0 \cdot t - \frac{1}{4} \frac{a_0^4 - 6a_0^2 c_0^2 - b_0^4 + c_0^4}{a_0^2 b_0 c_0^2} t^2 + \dots \\
 c(t) &= c_0 + 0 \cdot t - \frac{1}{4} \frac{a_0^4 - 6a_0^2 b_0^2 + b_0^4 - c_0^4}{a_0^2 b_0^2 c_0} t^2 + \dots \\
 f(t) &= 0 + 12t + 0 \cdot t^2 + \dots
 \end{aligned}$$

The next issue which we will discuss is how the smoothness conditions from Theorem 3.9 translate into conditions on the coefficients of the above power series. The modules $\mathfrak{p} = \mathbb{W}_{6,0} \oplus \mathbb{W}_{3,3} \oplus \mathbb{W}_{-3,3}$ and $\mathfrak{p}^\perp = \mathbb{W}_{6,0}$ have one submodule in common and Assumption 3.11 is thus not satisfied. Since we assume that the metric is diagonal, it is nevertheless an element of $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. We can therefore still check the smoothness conditions by describing W_m^h and W_m^v . $S^2(\mathfrak{p})$ decomposes as

$$\begin{aligned}
(61) \quad S^2(\mathfrak{p}) = & \mathbb{V}_{12,0} \oplus \mathbb{V}_{6,6} \oplus \mathbb{V}_{-6,6} \\
& \oplus \mathbb{V}_{9,3} \oplus \mathbb{V}_{3,-3} \oplus \mathbb{V}_{3,3} \oplus \mathbb{V}_{9,-3} \\
& \oplus \mathbb{V}_{0,6} \oplus \mathbb{V}_{6,0} \oplus 3\mathbb{R}
\end{aligned}$$

For $S^m(\mathfrak{p}^\perp)$, we obtain

$$(62) \quad S^m(\mathfrak{p}^\perp) = \begin{cases} \mathbb{V}_{6m,0} \oplus \mathbb{V}_{6(m-2),0} \oplus \dots \oplus \mathbb{V}_{12,0} \oplus \mathbb{R} & \text{if } m \text{ is even} \\ \mathbb{V}_{6m,0} \oplus \mathbb{V}_{6(m-2),0} \oplus \dots \oplus \mathbb{V}_{6,0} & \text{if } m \text{ is odd} \end{cases}$$

The dimensions of W_m^h and W_m^v can be calculated with the help of Schur's lemma.

$$(63) \quad \dim W_m^h = \begin{cases} 3 & \text{if } m = 0 \\ 5 & \text{if } m \geq 2 \text{ and even} \\ 2 & \text{if } m \text{ odd} \end{cases} \quad \dim W_m^v = \begin{cases} 1 & \text{if } m = 0 \\ 3 & \text{if } m \geq 2 \text{ and even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

The three dimensions of W_0^h correspond to the coefficients of $e^1 \otimes e^1 + e^2 \otimes e^2$, $e^3 \otimes e^3 + e^4 \otimes e^4$, and $e^5 \otimes e^5 + e^6 \otimes e^6$. The two additional dimensions of W_m^h where m is even and ≥ 2 are caused by the submodule $\mathbb{V}_{12,0}$ of $S^2(V_1)$. They therefore describe the coefficients of $e^1 \otimes e^1 - e^2 \otimes e^2$ and $e^1 \otimes e^2 + e^2 \otimes e^1$. Since we consider only diagonal metrics, we can ignore the freedom which we have for the second coefficient. The two dimensions of W_m^h where m is odd describe K -equivariant maps $S^m(\mathfrak{p}^\perp) \rightarrow S^2(\mathfrak{p})$ which are non-zero only on $V_2 \otimes V_3$ and can be ignored for the same reasons as above.

The two additional dimensions of W_m^v where m is even and ≥ 2 describe the freedom of $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ and $g(\frac{\partial}{\partial t}, e_7)$ and can be ignored, too (see Remark 3.10). As we have remarked at the same place, the value of $|f'(0)|$ has to be chosen in such a way that the length of the collapsing circle is $2\pi t + O(t^2)$ for small t . Since $\exp(e_7 t)$ intersects $U(1)_{1,1}$ at $t = \frac{\pi}{3}$ for the first time, the length of the collapsing circle is

$$\int_0^{\frac{\pi}{3}} \sqrt{g(e_7, e_7)} ds = \frac{\pi}{3} |f(t)|.$$

We therefore obtain $|f'(0)| = 6$ as a smoothness condition. By translating our statements on W_m^h and W_m^v into conditions on power series expansion of the metric we finally see that an analytic diagonal metric of type (22) is smooth if and only if

- $a_1^2(0) = a_2^2(0)$,
- a_1^2, a_2^2, b^2 , and c^2 are even functions,
- $f(0) = 0$, $|f'(0)| = 6$, and

- f is odd.

The power series (60) does not satisfy these conditions. Nevertheless, there is a method to obtain smooth cohomogeneity-one metrics whose holonomy is contained in $\text{Spin}(7)$ from (60). Let

$$(64) \quad h := \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SU(3) .$$

h acts by $h.gU(1)_{1,1} := hgh^{-1}U(1)_{1,1}$ on $N^{1,1}$ and stabilizes the G_2 -structures which are determined by a basis of type (27). Since $h^2 \in U(1)_{1,1}$, $N^{1,1}$ is a double cover of the quotient of $SU(3)$ by the group which is generated by $U(1)_{1,1}$ and h . We denote this quotient simply by $N^{1,1}/\mathbb{Z}_2$. On $SU(3)/U(1)^2$, h acts trivially. We divide the cohomogeneity-one manifold with principal orbit $N^{1,1}$ and singular orbit $SU(3)/U(1)^2$ by the group which is generated by the simultaneous action of h on all orbits. Any (smooth or non-smooth) $\text{Spin}(7)$ -structure on the old manifold which is induced by (27) is mapped by the quotient map to a new one.

It is easy to see that any circle which was wrapped r times around the origin of the old normal space is wrapped $2r$ times around the origin of the new normal space. We reconsider the arguments which we have made and see that in this new situation we have to require $|f'(0)| = 12$ instead of $|f'(0)| = 6$ in order to make the metric smooth at the singular orbit. Since $U(1)^2$ now acts on \mathfrak{p}^\perp as $\mathbb{V}_{12,0}$ rather than $\mathbb{V}_{6,0}$, we have

$$(65) \quad \dim W_m^h = \begin{cases} 3 & \text{if } m \text{ even} \\ 2 & \text{if } m \text{ odd} \end{cases}$$

Since the values of $\dim W_m^h$ have changed, the smoothness conditions on the functions a_1 , a_2 , b , and c have changed, too. The meaning of the three dimensions in the even case is the same as before and the two dimensions in the odd case now correspond to the components of the metric in $S^2(V_1)$. The dimensions of W_m^v stay the same and we have found the following new smoothness conditions.

- $a_1^2(t) = a_2^2(-t)$,
- b^2 and c^2 are even,
- $f(0) = 0$, $|f'(0)| = 12$, and
- f is odd.

In particular, these conditions are satisfied if $a_1(t) = -a_2(-t)$, $b(t) = b(-t)$ and $c(t) = c(-t)$. The first coefficients of (60) obviously satisfy this new set of conditions.

By an explicit calculation we can prove that any power series solution of (28) with the same initial conditions as in this subsection is uniquely determined by a_0 , b_0 , and c_0 . The system (28) is preserved if we replace $(a_1(t), a_2(t), b(t), c(t), f(t))$ by $(-a_2(-t), -a_1(-t), b(-t), c(-t), -f(-t))$. Therefore, both sets of functions are the unique solutions of the same initial value problem and the power series thus satisfies the smoothness conditions. h acts trivially on \mathfrak{p}^\perp and non-trivially on V_1 , V_2 , and V_3 . For this reason, Assumption 3.11 is satisfied and we can show with the help of Theorem 3.12 that the series converges near the singular orbit. Moreover, we can use the dimensions of W_m^h and W_m^v to calculate the number of cohomogeneity-one Einstein metrics near the singular orbit.

We take another look at the power series (60). The condition $a_1(t) + a_2(t) = 0$ can only be satisfied if $a_0^2 = b_0^2 + c_0^2$. In that situation, we obtain the metrics of Bazaikin and Malkovich [6], which have holonomy $SU(4)$. We finally have proven the following theorem.

Theorem 5.3. *Let M be a cohomogeneity-one manifold whose principal orbit is $N^{1,1}/\mathbb{Z}_2$ where \mathbb{Z}_2 is generated by (64). We assume that M has exactly one singular orbit of type $SU(3)/U(1)^2$, which is at $t = 0$.*

- (1) *For any $a_0, b_0, c_0 \in \mathbb{R} \setminus \{0\}$ there exists a unique $SU(3)$ -invariant parallel $Spin(7)$ -structure Ω on a tubular neighborhood of $SU(3)/U(1)^2$ which is determined by a basis of type (27) and satisfies $f(0) = 0$, $a_1(0) = -a_2(0) = a_0$, $b(0) = b_0$, and $c(0) = c_0$.*
- (2) *The metric associated to Ω is a diagonal metric of type (22). If $a_0^2 = b_0^2 + c_0^2$, its holonomy is $SU(4)$.*
- (3) *For any choice of $a_0, b_0, c_0 \in \mathbb{R} \setminus \{0\}$ and $a_1, \beta, f_3, \lambda \in \mathbb{R}$, there exists a unique $SU(3)$ -invariant Einstein metric on a tubular neighborhood of $SU(3)/U(1)^2$ such that*
 - (a) $f(0) = 0$, $a_1(0)^2 = a_2(0)^2 = a_0^2$, $b(0)^2 = b_0^2$, $c(0)^2 = c_0^2$,
 - (b) $(a_1 - a_2)'(0) = a_1$, $\beta'_{1,2}(0) = \beta$,
 - (c) $f'''(0) = f_3$, and
 - (d) *the Einstein constant is λ .*

Remark 5.4. A power series ansatz for the metrics with special holonomy from the above theorem has also been made in Kanno, Yasui [23]. These metrics were also investigated by Bazaikin and Malkovich [4], [6]. The authors study the smoothness and completeness of the metrics and describe the family of metrics with holonomy $SU(4)$ explicitly. They also prove that the holonomy of their metrics is always $SU(4)$ except in the limiting case where $b_0 \rightarrow 0$ and the metric degenerates into the hyperkähler metric of Calabi [10] on $T^*\mathbb{CP}^2$. Our methods for the calculation of the smoothness conditions and for the proof of the holonomy reduction in the case $a_1(t) + a_2(t) = 0$ are new contributions of the author.

6. S^5 AS SINGULAR ORBIT

S^5 is a possible singular orbit only if $k \cdot l \cdot (-k - l) = 0$. We assume in this section that $(k, l) = (1, -1)$, since this implies the initial conditions $a(0) = 0$ and $b(0), c(0), f(0) \neq 0$. The only difference to the case $(k, l) = (1, 0)$ is that the off-diagonal coefficients of the metric are contained in $V_2 \otimes V_3 \oplus V_3 \otimes V_2$ instead of $V_1 \otimes V_3 \oplus V_3 \otimes V_1$ and hence will be denoted by $\beta_{3,5}$ and $\beta_{3,6}$ instead of $\beta_{1,5}$ and $\beta_{1,6}$. We want the right-hand side of the second and third equation of (26) to converge to a real number for $t \rightarrow 0$. This is only possible if $b(0)^2 = c(0)^2$. Since we may replace c by $-c$, f by $-f$, and t by $-t$ without changing the system (26), we can assume that $b(0) = c(0)$. We shortly denote $b(0)$ by b_0 and $f(0)$ by f_0 . By a power series ansatz, we obtain the following solution of the system (26).

(66)

$$\begin{aligned}
a(t) &= 0 + 2t + 0 \cdot t^2 - \frac{1}{27} \frac{36b_0^2 - f_0^2}{b_0^4} t^3 + \dots \\
b(t) &= b_0 - \frac{1}{6} \frac{f_0}{b_0} t + \frac{1}{72} \frac{72b_0^2 - 5f_0^2}{b_0^3} t^2 + \frac{1}{6480} \frac{f_0(504b_0^2 - 167f_0^2)}{b_0^5} t^3 + \dots \\
c(t) &= b_0 + \frac{1}{6} \frac{f_0}{b_0} t + \frac{1}{72} \frac{72b_0^2 - 5f_0^2}{b_0^3} t^2 - \frac{1}{6480} \frac{f_0(504b_0^2 - 167f_0^2)}{b_0^5} t^3 + \dots \\
f(t) &= f_0 + 0 \cdot t + \frac{1}{6} \frac{f_0^3}{b_0^4} t^2 + 0 \cdot t^3 + \dots
\end{aligned}$$

Our next step is to decompose $S^m(\mathfrak{p}^\perp)$ and $S^2(\mathfrak{p})$ into $SU(2)$ -submodules. This is necessary in order to deduce the smoothness conditions. We denote the complex irreducible representation of $SU(2)$ with weight r by $\mathbb{V}_r^{\mathbb{C}}$. The real irreducible representation with the same weight we denote by $\mathbb{V}_r^{\mathbb{R}}$. We recall that $\dim \mathbb{V}_r^{\mathbb{R}} = r + 1$ if r is even and $\dim \mathbb{V}_r^{\mathbb{R}} = 2(r + 1)$ if r is odd.

The orbits of the $SU(2)$ -action on the three-dimensional space \mathfrak{p}^\perp are spheres. Therefore, it is isomorphic to $\mathbb{V}_2^{\mathbb{R}}$. In order to decompose $S^m(\mathfrak{p}^\perp)$, we first consider the space $S_{\mathbb{R}}^m(\mathbb{C}^2)$ of all homogeneous polynomials of m^{th} order with real coefficients on \mathbb{C}^2 . The subscript \mathbb{R} means that we consider \mathbb{C}^2 as a four-dimensional real vector space and we have $\dim S_{\mathbb{R}}^2(\mathbb{C}^2) = 10$, $\dim S_{\mathbb{R}}^3(\mathbb{C}^2) = 20$, etc. The complexification $S_{\mathbb{R}}^m(\mathbb{C}^2) \otimes \mathbb{C}$ consists of all polynomials depending on $z_1, z_2, \overline{z_1}, \overline{z_2}$. Let $\mathbb{V}_{s, \overline{m-s}}$ be the subspace of all trace-free polynomials depending on s complex and $m - s$ conjugate complex variables. $S_{\mathbb{R}}^m(\mathbb{C}^2) \otimes \mathbb{C}$ and $\mathbb{V}_{s, \overline{m-s}}$ are obviously $SU(2)$ -modules and we have the following decomposition

$$(67) \quad S_{\mathbb{R}}^m(\mathbb{C}^2) \otimes \mathbb{C} = \bigoplus_{p=0}^{\lfloor \frac{m}{2} \rfloor} \bigoplus_{s=0}^{m-2p} \mathbb{V}_{s, \overline{m-2p-s}}.$$

The submodules $\mathbb{V}_{s, \overline{m-2p-s}}$ are irreducible and by calculating their dimension we see that

$$(68) \quad S_{\mathbb{R}}^m(\mathbb{C}^2) \otimes \mathbb{C} = \bigoplus_{p=0}^{\lfloor \frac{m}{2} \rfloor} (m-2p+1) \mathbb{V}_{m-2p}^{\mathbb{C}},$$

where $(m-2p+1) \mathbb{V}_{m-2p}^{\mathbb{C}}$ denotes the direct sum of $m-2p+1$ copies of $\mathbb{V}_{m-2p}^{\mathbb{C}}$. $S_{\mathbb{R}}^m(\mathbb{C}^2)$ consists of exactly those elements of $S_{\mathbb{R}}^m(\mathbb{C}^2) \otimes \mathbb{C}$ which are invariant with respect to the conjugation map τ . τ maps $\mathbb{V}_{s, \overline{m-2p-s}}$ into $\mathbb{V}_{m-2p-s, \bar{s}}$ and vice versa. If m is even and $s \neq \frac{m-2p}{2}$, the subspace of $\mathbb{V}_{s, \overline{m-2p-s}} \oplus \mathbb{V}_{m-2p-s, \bar{s}}$ is invariant under τ and decomposes into two real submodules of the same dimension. Both submodules are irreducible and equivalent to $\mathbb{V}_{m-2p}^{\mathbb{R}}$. $\mathbb{V}_{\frac{m-2p}{2}, \overline{\frac{m-2p}{2}}}$ is a real module which is also isomorphic to $\mathbb{V}_{m-2p}^{\mathbb{R}}$. If m is odd, the subspace of $\mathbb{V}_{s, \overline{m-2p-s}} \oplus \mathbb{V}_{m-2p-s, \bar{s}}$ is invariant under τ , irreducible, and equivalent to $\mathbb{V}_{m-2p}^{\mathbb{R}}$. All in all, we have

$$(69) \quad S_{\mathbb{R}}^m(\mathbb{C}^2) = \begin{cases} \bigoplus_{p=0}^{\frac{m}{2}} (2p+1) \mathbb{V}_{2p}^{\mathbb{R}} & \text{if } m \text{ is even} \\ \bigoplus_{p=0}^{\frac{m-1}{2}} (p+1) \mathbb{V}_{2p+1}^{\mathbb{R}} & \text{if } m \text{ is odd} \end{cases}$$

$\mathbb{V}_2^{\mathbb{R}}$ can be identified with $\mathbb{V}_{1, \bar{1}}$. $S^m(\mathbb{V}_2^{\mathbb{R}})$ therefore is a submodule of $\mathbb{V}_{m, \bar{m}}$. With the help of the above considerations we obtain

$$(70) \quad S^m(\mathbb{V}_2^{\mathbb{R}}) = \bigoplus_{p=0}^{\lfloor \frac{m}{2} \rfloor} \mathbb{V}_{m-2p, \overline{m-2p}} = \bigoplus_{p=0}^{\lfloor \frac{m}{2} \rfloor} \mathbb{V}_{2m-4p}^{\mathbb{R}}.$$

We are now able to calculate the dimension of W_m^h and W_m^v . By a short calculation we see that

$$(71) \quad \mathfrak{p} = \mathbb{V}_1^{\mathbb{C}} \oplus \mathbb{V}_0^{\mathbb{R}}.$$

With the help of (69) we conclude that

$$(72) \quad S^2(\mathfrak{p}) = S^2(\mathbb{V}_1^{\mathbb{C}}) \oplus (\mathbb{V}_1^{\mathbb{C}} \otimes \mathbb{V}_0^{\mathbb{R}}) \oplus S^2(\mathbb{V}_0^{\mathbb{R}}) = 3\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_1^{\mathbb{C}} \oplus 2\mathbb{V}_0^{\mathbb{R}}$$

and finally obtain

$$(73) \quad \dim W_m^h = \begin{cases} 2 & \text{if } m \text{ is even} \\ 3 & \text{if } m \text{ is odd} \end{cases}$$

We interpret these numbers and start with the case $m = 0$. Any $SU(2)$ -invariant metric on the singular orbit is diagonal and satisfies $b(0)^2 = c(0)^2$. $\dim W_0^h = 2$ therefore simply means that we can choose the initial values b_0 and f_0 freely. Let b_m , c_m , and f_m denote the m^{th} coefficients of the power series for b , c , and f . Analogously to the case $m = 0$, $\dim W_m^h = 2$ for even m means that $b_m = c_m$.

There is a suitable $h \in SU(2)$ which acts on $\text{span}(e_3, e_4, e_5, e_6) \subseteq \mathfrak{p}$ in the same way as $j \in Sp(1)$ by right-multiplication on \mathbb{H} . Since \mathfrak{p}^\perp is three-dimensional, j acts as a rotation around an angle of π . Moreover, it is a rotation around an axis perpendicular to $\frac{\partial}{\partial t}$ and thus turns $\frac{\partial}{\partial t}$ into $-\frac{\partial}{\partial t}$. Since the metric g is invariant under h , it follows that

$$\begin{aligned} g_t(e_3, e_3) &= g_{-t}(e_5, e_5) \\ g_t(e_3, e_5) &= -g_{-t}(e_5, e_3) \\ g_t(e_3, e_6) &= -g_{-t}(e_5, e_4) = -g_{-t}(e_3, e_6) \\ g_t(e_7, e_7) &= g_{-t}(e_7, e_7) \end{aligned}$$

This translates into $b_m = -c_m$ and $f_m = 0$ if m is odd. The space of all $U(1)_{1,-1}$ -invariant elements of $S^2(\mathfrak{p})$ satisfying these conditions has dimension 3 which equals $\dim W_m^h$. Therefore, there are no further conditions on the horizontal part and the horizontal part of an analytic metric is smooth if and only if

$$(74) \quad b(t) = c(-t) \quad \text{and} \quad f, \beta_{3,5}, \beta_{3,6} \text{ are even.}$$

We turn to the vertical component of the metric, which is determined by a . It follows from (70) that

$$(75) \quad S^2(\mathfrak{p}^\perp) = \mathbb{V}_4^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}}.$$

Since $\dim \text{Hom}_{SU(2)}(\mathbb{V}_4^{\mathbb{R}}, \mathbb{V}_4^{\mathbb{R}}) = 1$, we have

$$(76) \quad \dim W_m^v = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \geq 2 \text{ and even} \end{cases}$$

For similar reasons as in Subsection 5.3, this means that a has to be an odd function. The only missing smoothness condition is that the length $\ell(t)$ of any great circle of the collapsing sphere $SU(2)/U(1)_{1,-1}$ has to be $2\pi t + O(t^2)$ for small t . The Lie group which is generated by e_1 intersects $U(1)_{1,-1}$ twice. Therefore, $\ell(t) = \sqrt{g_t(e_1, e_1)} \pi = |a(t)|\pi$. By the same argument as in Subsection 5.3, it follows that $|a'(0)|$ has to be 2, which is indeed the case.

As in the previous case, we make an explicit calculation and prove that for any choice of $b_0, f_0 \in \mathbb{R} \setminus \{0\}$ there exists a unique power series solution of (26). $(a(t), b(t), c(t), f(t)) \mapsto (-a(-t), c(-t), b(-t), f(-t))$ is a symmetry of (26) if $(k, l) = (1, -1)$. As in the previous subsection, it follows that the power series satisfies the smoothness conditions.

Unfortunately, Assumption 3.11 is not satisfied, since \mathfrak{p} and \mathfrak{p}^\perp both contain a trivial submodule, namely $\text{span}(e_7)$ or $\text{span}(\frac{\partial}{\partial t})$ respectively. Nevertheless, we always have due to the choice of our coordinates $g(e_7, \frac{\partial}{\partial t}) = 0$. Moreover, we have $\text{Ric}(e_7, \frac{\partial}{\partial t}) = 0$ (see Grove, Ziller [19]). Therefore, g , Ric , and the steps of the Picard iteration in Eschenburg, Wang [15] do not leave the space $S^2(\mathfrak{p}) \oplus S^2(\mathfrak{p}^\perp)$. The arguments of the proof in [15] therefore remain unchanged and we can conclude that the power series converges and are able to determine the number of the Einstein metrics.

Theorem 6.1. *Let M be a cohomogeneity-one manifold whose principal orbit is $N^{1,-1}$. We assume that M has exactly one singular orbit of type S^5 , which is at $t = 0$.*

- (1) *In this situation, any cohomogeneity-one metric satisfies $a(0) = 0$.*
- (2) *For any $b_0, f_0 \in \mathbb{R} \setminus \{0\}$ there exists a unique $SU(3)$ -invariant parallel $\text{Spin}(7)$ -structure Ω on a tubular neighborhood of S^5 which is determined by a basis of type (25) and satisfies $b(0) = c(0) = b_0$ and $f(0) = f_0$. The holonomy of the associated metric is all of $\text{Spin}(7)$.*
- (3) *For any choice of $b_0, f_0 \in \mathbb{R} \setminus \{0\}$ and $b_1, \beta, \tilde{\beta}, a_3, \lambda \in \mathbb{R}$, there exists a unique $SU(3)$ -invariant Einstein metric on a tubular neighborhood of S^5 such that*
 - (a) $b(0)^2 = c(0)^2 = b_0^2, f(0)^2 = f_0^2,$
 - (b) $(b - c)'(0) = b_1, \beta'_{3,5}(0) = \beta, \beta'_{3,6}(0) = \tilde{\beta},$
 - (c) $a'''(0) = a_3,$ and
 - (d) *the Einstein constant is λ .*

Remark 6.2. (1) The metrics with singular orbit S^5 and holonomy $\text{Spin}(7)$ were also considered by Cvetič et al. [13]. The discussion of

the smoothness conditions, the convergence of the power series and the existence of the Einstein metrics are new results.

- (2) If the principal orbit is $N^{1,-1}$, the singular orbit can also be a space of type $SU(3)/SO(3)$. However, it is impossible that the metric on $SU(3)/SO(3)$ is positive and the volume of $SO(3)/U(1)_{1,-1}$ shrinks to zero if the metric is diagonal with respect to $(e_i)_{i=1 \leq i \leq 7}$. Although it is possible to construct spaces with singular orbit $SU(3)/SO(3)$ by considering non-diagonal metrics, we will not investigate this case further.

7. \mathbb{CP}^2 AS SINGULAR ORBIT

We assume that the principal orbit is a generic Aloff-Wallach space $N^{k,l}$. In this situation, the isotropy algebra \mathfrak{k} of the $SU(3)$ -action on \mathbb{CP}^2 is either $\mathfrak{u}(1)_{k,l} \oplus V_1 \oplus V_4$, $\mathfrak{u}(1)_{k,l} \oplus V_2 \oplus V_4$, or $\mathfrak{u}(1)_{k,l} \oplus V_3 \oplus V_4$. As we have remarked in Convention 4.3, it suffices to work with the case $\mathfrak{k} = \mathfrak{u}(1)_{k,l} \oplus V_1 \oplus V_4$ if we consider all pairs (k, l) of coprime integers with $k \geq l$.

If $(k, l) = (1, -1)$, $\mathfrak{u}(1)_{k,l} \oplus V_1 \oplus V_4$ is not a possible choice of \mathfrak{k} since $K/U(1)_{1,-1}$ is diffeomorphic to $S^2 \times S^1$. Nevertheless, the two related cases $(k, l) \in \{(1, 0), (0, -1)\}$ are still possible.

If $k = l = 1$ and the metric is diagonal, $\mathfrak{u}(1)_{1,1} \oplus V_1 \oplus V_4$, $\mathfrak{u}(1)_{1,1} \oplus V_2 \oplus V_4$, and $\mathfrak{u}(1)_{1,1} \oplus V_3 \oplus V_4$ are still the only possibilities for \mathfrak{k} . If the complement of $\mathfrak{u}(1)_{1,1}$ was transversely embedded into $V_1 \oplus V_2 \oplus V_3$, the null space of the degenerate metric $g_0 \in S^2(\mathfrak{m})$ could not be \mathfrak{k} . Moreover, $\mathfrak{u}(1)_{1,1} \oplus V_2 \oplus V_4$ and $\mathfrak{u}(1)_{1,1} \oplus V_3 \oplus V_4$ can be obtained by the action of an element of $\text{Norm}_{SU(3)}U(1)_{1,1}$ from each other. We therefore have to consider only one of these cases. All in all, we have to consider the following three initial values problems.

- (1) The system (26) with $a(0) = f(0) = 0$ and $(k, l) \notin \{(1, -1), (1, 1), (1, -2), (2, -1)\}$,
- (2) The system (28) with $a_1(0) = a_2(0) = f(0) = 0$,
- (3) The system (28) with $b(0) = f(0) = 0$.

We make a power series ansatz for all of the above initial value problems and start with the first one. As in Section 6, we can assume without loss of generality that $b(0) = c(0) =: b_0$. The Taylor expansion of any solution of our initial value problem begins with

$$\begin{aligned}
(77) \quad a(t) &= 0 + t + 0 \cdot t^2 - \frac{1}{24b_0^2} \frac{12\Delta + q(k+l)}{\Delta} t^3 + 0 \cdot t^4 + \dots \\
b(t) &= b_0 + 0 \cdot t + \frac{1}{6b_0} \frac{4k+5l}{k+l} t^2 + 0 \cdot t^3 \\
&\quad + \frac{1}{288b_0^3} \frac{(-104k^2 - 224kl - 140l^2)\Delta + q(-k^3 - k^2l + kl^2 + l^3)}{\Delta^2} t^4 + \dots \\
c(t) &= b_0 + 0 \cdot t + \frac{1}{6b_0} \frac{5k+4l}{k+l} t^2 + 0 \cdot t^3 \\
&\quad + \frac{1}{288b_0^3} \frac{(-140k^2 - 224kl - 104l^2)\Delta + q(k^3 + k^2l - kl^2 - l^3)}{\Delta^2} t^4 + \dots \\
f(t) &= 0 + \frac{2\Delta}{k+l} t + 0 \cdot t^2 + \frac{q}{6b_0^2} t^3 + 0 \cdot t^4 + \dots
\end{aligned}$$

The parameter q of third order can be chosen freely. Any solution of (28) with $a_1(0) = a_2(0) = f(0)$ also has to satisfy $b(0)^2 = c(0)^2$. We again can assume that $b(0) = c(0) =: b_0$, since $(c(t), f(t)) \mapsto (-c(-t), -f(-t))$ is a symmetry of the system (28), and obtain the following Taylor expansion.

$$\begin{aligned}
(78) \quad a_1(t) &= 0 + t + 0 \cdot t^2 + \frac{q_1}{6b_0^2} t^3 + 0 \cdot t^4 \\
&\quad + \frac{2q_2^1 - 3q_1q_2 - 3q_2^2 - 3q_1 - 18q_2}{60b_0^4} t^5 + \dots \\
a_2(t) &= 0 + t + 0 \cdot t^2 + \frac{q_2}{6b_0^2} t^3 + 0 \cdot t^4 \\
&\quad + \frac{-3q_1^2 - 3q_1q_2 + 2q_2^2 - 18q_1 - 3q_2}{60b_0^4} t^5 + \dots \\
b(t) &= b_0 + 0 \cdot t + \frac{3}{4b_0} t^2 + 0 \cdot t^3 - \frac{39}{96b_0^3} t^4 + 0 \cdot t^5 + \dots \\
c(t) &= b_0 + 0 \cdot t + \frac{3}{4b_0} t^2 + 0 \cdot t^3 - \frac{39}{96b_0^3} t^4 + 0 \cdot t^5 + \dots \\
f(t) &= 0 + 3t + 0 \cdot t^2 - \frac{6+q_1+q_2}{2b_0^2} t^3 + 0 \cdot t^4 \\
&\quad + \frac{2q_1^2 + 7q_1q_2 + 2q_2^2 + 27q_1 + 27q_2 + 90}{20b_0^4} t^5 + \dots
\end{aligned}$$

Analogously to the previous case, there are two parameters q_1 and q_2 which can be chosen freely. The functions b and c coincide up to fifth order. Later on, we will prove that actually $b(t) = c(t)$ for all values of t . Next we consider the equations (28) under the assumption that $b(0) = f(0) = 0$.

We necessarily have $c(0)^2 = a_1(0)^2 = a_2(0)^2$. Let $a_0 := a_1(0)$. Since there are four possibilities for the signs of $a_2(0)$ and $c(0)$, there are four kinds of initial value problems. Because of the symmetry of (28) which we have used in the previous case, we can assume that the sign of $a_1(0)$ and $c(0)$ is the same. Therefore, the only two subcases which we have to consider are $a_1(0) = a_2(0)$ and $a_1(0) = -a_2(0)$. The initial condition $a_1(0) = a_2(0)$ yields the following power series.

$$\begin{aligned}
(79) \quad a_1(t) &= a_0 + 0 \cdot t + \frac{1}{a_0} t^2 + 0 \cdot t^3 - \frac{q+21}{24a_0^3} t^4 + 0 \cdot t^5 + \dots \\
a_2(t) &= a_0 + 0 \cdot t + \frac{1}{a_0} t^2 + 0 \cdot t^3 - \frac{q+21}{24a_0^3} t^4 + 0 \cdot t^5 + \dots \\
b(t) &= 0 + t + 0 \cdot t^2 + \frac{q}{6a_0^2} t^3 + 0 \cdot t^4 \\
&\quad - \frac{8q^2+42q+9}{120a_0^4} t^5 + \dots \\
c(t) &= a_0 + 0 \cdot t + \frac{1}{2a_0} t^2 + 0 \cdot t^3 + \frac{q-6}{24a_0^3} t^4 + 0 \cdot t^5 + \dots \\
f(t) &= 0 - 6t + 0 \cdot t^2 + \frac{2(q+3)}{a_0^2} t^3 + 0 \cdot t^4 \\
&\quad - \frac{11q^2+54q+123}{10a_0^4} t^5 + \dots
\end{aligned}$$

where q is a free parameter. Later on it will be proven that $a_1(t) = a_2(t)$ for all t . Next, we study (28) under the assumption that $b(0) = f(0) = 0$ and $a_1(0) = -a_2(0)$. We obtain a system of quadratic equations for the first derivatives $(a_1'(0), \dots, f'(0))$. The only two meaningful solutions of that system are $(0, 0, 1, 0, 6)$ and $(0, 0, -1, 0, 6)$. $(a_1(t), a_2(t), b(t), c(t), f(t)) \mapsto (-a_2(-t), -a_1(-t), b(-t), c(-t), -f(-t))$ is another symmetry of (28). Since it maps a solution with $b'(0) = 1$ into a solution with $b'(0) = -1$, we only need to consider the case $b'(0) = 1$ and obtain

$$\begin{aligned}
a_1(t) &= a_0 + 0 \cdot t + \frac{1}{a_0} t^2 + 0 \cdot t^3 + \frac{3q-23}{24a_0^3} t^4 \\
&\quad + 0 \cdot t^5 + \dots \\
a_2(t) &= -a_0 + 0 \cdot t + \frac{q-2}{a_0} t^2 + 0 \cdot t^3 - \frac{12q^2-25q-7}{24a_0^3} t^4 \\
&\quad + 0 \cdot t^5 + \dots \\
b(t) &= 0 + t + 0 \cdot t^2 - \frac{1}{6a_0^2} t^3 + 0 \cdot t^4 \\
(80) \quad &\quad - \frac{39q^2-114q+25}{240a_0^4} t^5 + \dots \\
c(t) &= a_0 + 0 \cdot t + \frac{q}{2a_0} t^2 + 0 \cdot t^3 + \frac{3q^2-13q+3}{24a_0^3} t^4 \\
&\quad + 0 \cdot t^5 + \dots \\
f(t) &= 0 + 6t + 0 \cdot t^2 - \frac{4}{a_0^2} t^3 + 0 \cdot t^4 \\
&\quad + \frac{3q^2-18q+175}{20a_0^4} t^5 + \dots
\end{aligned}$$

As usual, q can be chosen freely. Our aim is to prove that the three initial value problems have a unique smooth solution for any choice of the parameters a_0 , b_0 , q , q_1 , and q_2 . We therefore have to check the smoothness conditions for the above power series. In [30], we have proven that an analytic diagonal metric $g = g_t + dt^2$ of cohomogeneity one has a smooth extension to a singular orbit at $t = 0$ if

- (1) g_t converges for $t \rightarrow 0$ to a degenerate bilinear form which is invariant with respect to the cohomogeneity-one action.
- (2) The sectional curvature of the collapsing sphere behaves as $\frac{1}{t} + O(1)$ for $t \rightarrow 0$.
- (3) The coefficient functions of the horizontal part are even.
- (4) The coefficient functions of the vertical part are odd.

We remark that this result can also be applied to orbifold metrics. The metric on the singular orbit \mathbb{CP}^2 is in all three cases the Fubini study metric and thus $SU(3)$ -invariant.

In order to check the second smoothness condition, which we have also mentioned in Remark 3.10, we have to search for the metric h with constant sectional curvature 1 on $K/U(1)_{k,l}$. Let \tilde{h} with $\tilde{h}(X, Y) = -\frac{1}{2}\text{tr}(XY)$ for all $X, Y \in \mathfrak{su}(2)$ be the metric with sectional curvature 1 on $SU(2)$. We embed $SU(2)$ into $SU(3)$ such that its Lie algebra becomes $\mathfrak{u}(1)_{1,-1} \oplus V_1$. The map

$\pi : SU(2) \rightarrow K/U(1)_{k,l}$ with $\pi(k) := kU(1)_{k,l}$ is a covering map. h therefore has to satisfy

$$(81) \quad \|d\pi(X)\|_h = \|X\|_{\tilde{h}}.$$

With the help of this formula, we see that

- (1) in the case where $a(0) = f(0) = 0$, $\|e_1\|_q = \|e_2\|_q = 1$ and $\|e_7\|_q = |\frac{2\Delta}{k+l}|$. We therefore obtain $|a'(0)| = 1$ and $|f'(0)| = |\frac{2\Delta}{k+l}|$.
- (2) in the case where $a_1(0) = a_2(0) = f(0) = 0$, $\|e_1\|_q = \|e_2\|_q = 1$ and $\|e_7\|_q = 3$. We therefore obtain $|a'_1(0)| = |a'_2(0)| = 1$ and $|f'(0)| = 3$.
- (3) in the case where $b(0) = f(0) = 0$, $\|e_3\|_q = \|e_4\|_q = 1$ and $\|e_7\|_q = 6$. We therefore obtain $|b'(0)| = 1$ and $|f'(0)| = 6$.

Here, q denotes the biinvariant metric on $\mathfrak{su}(3)$ with $q(X, Y) = -\frac{1}{2}\text{tr}(XY)$ which we have introduced earlier.

The power series (77), (78), (79), and (80) obviously satisfy the first two smoothness conditions. As in the previous two sections, we can prove that the initial value problems have a unique power series solution for any choice of a_0 , b_0 , q , q_1 , and q_2 . We are now able to prove the remaining smoothness conditions by means of symmetry arguments.

- (1) $(a(t), b(t), c(t), f(t)) \mapsto (-a(-t), b(-t), c(-t), -f(-t))$ is a symmetry of (26). Any solution of that system with $a(0) = f(0) = 0$, $b(0) = c(0) = b_0$, and $f'''(0) = \frac{q}{b_0^2}$ is mapped by the symmetry to another solution with the same initial values. a and f therefore are odd functions and b and c are even. The power series (77) thus satisfies all smoothness conditions.
- (2) The smoothness of (78) can be proven with the help of the symmetry $(a_1(t), a_2(t), b(t), c(t), f(t)) \mapsto (-a_1(-t), -a_2(-t), b(-t), c(-t), -f(-t))$ of (28). The relation $b(t) = c(t)$ follows with the help of the symmetry $(b, c) \mapsto (c, b)$.
- (3) The smoothness of (79) can be proven with the help of the symmetry $(a_1(t), a_2(t), b(t), c(t), f(t)) \mapsto (a_1(-t), a_2(-t), -b(-t), c(-t), -f(-t))$ of (28) and the relation $a_1(t) = a_2(t)$ follows with the help of the symmetry $(a_1, a_2) \mapsto (a_2, a_1)$.
- (4) The smoothness of (80) can be proven with the help of the symmetry $(a_1(t), a_2(t), b(t), c(t), f(t)) \mapsto (a_2(-t), -a_1(-t), -b(-t), c(-t), -f(-t))$ of (28).

We finally have to prove that the power series converge. In the setting of this section, Assumption 3.11 is not always satisfied. If $k = l = 1$ and $b(0) = f(0) = 0$, \mathfrak{p} and \mathfrak{p}^\perp both contain a trivial $U(1)_{1,1}$ -submodule. The spaces \mathfrak{p} and \mathfrak{p}^\perp also contain a common submodule if $k = 1$, $l = 0$, and

$a(0) = f(0) = 0$. We can nevertheless prove the convergence with the help of the arguments which we have made in Remark 3.13.5.

Since 3.11 is in some cases not satisfied and we have not described the spaces W_m^h and W_2^v explicitly, we will not study the existence of Einstein metrics on a tubular simultaneous of \mathbb{CP}^2 . We conclude this section by summarizing our results on metrics with special holonomy.

Theorem 7.1. *Let M be a cohomogeneity-one orbifold whose principal orbit is $N^{k,l}$. We assume that M has exactly one singular orbit of type \mathbb{CP}^2 , which is at $t = 0$.*

- (1) *Let $N^{k,l}$ be not $SU(3)$ -equivariantly diffeomorphic to $N^{1,1}$ and let $k+l \neq 0$. For any $b_0 \in \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{R}$ there exists a unique $SU(3)$ -invariant parallel $Spin(7)$ -structure Ω on a tubular simultaneous of \mathbb{CP}^2 which is determined by a basis of type (25) and satisfies $a(0) = f(0) = 0$, $b(0) = c(0) = b_0$, and $f'''(0) = \frac{q}{b_0^2}$. The holonomy of the associated metric is all of $Spin(7)$.*
- (2) *Let $k = l = 1$. For any $b_0 \in \mathbb{R} \setminus \{0\}$ and $q_1, q_2 \in \mathbb{R}$ there exists a unique $SU(3)$ -invariant parallel $Spin(7)$ -structure Ω on a tubular simultaneous of \mathbb{CP}^2 which is determined by a basis of type (27) and satisfies $a_1(0) = a_2(0) = f(0) = 0$, $b(0) = c(0) = b_0$, $a_1'''(0) = \frac{q_1}{b_0^2}$ and $a_2'''(0) = \frac{q_2}{b_0^2}$. Moreover, we have $b(t) = c(t)$ for all values of t .*
- (3) *Let $k = l = 1$. For any $a_0 \in \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{R}$ there exists a unique $SU(3)$ -invariant parallel $Spin(7)$ -structure Ω on a tubular simultaneous of \mathbb{CP}^2 which is determined by a basis of type (27) and satisfies $b(0) = f(0) = 0$, $a_1(0) = a_2(0) = c(0) = a_0$, and $b'''(0) = \frac{q}{a_0^2}$. Moreover, we have $a_1(t) = a_2(t)$ for all values of t .*
- (4) *Let $k = l = 1$. For any $a_0 \in \mathbb{R} \setminus \{0\}$ and $q \in \mathbb{R}$ there exists a unique $SU(3)$ -invariant parallel $Spin(7)$ -structure Ω on a tubular simultaneous of \mathbb{CP}^2 which is determined by a basis of type (27) and satisfies $b(0) = f(0) = 0$, $a_1(0) = -a_2(0) = c(0) = a_0$, $b'(0) = 1$, and $c''(0) = \frac{q}{a_0}$.*

Remark 7.2. (1) The first class of metrics from the above theorem was considered by Cvetič [13] and by Kanno, Yasui [23]. The second class of metrics was discovered independently of the author by Bazaikin [5]. We have interpreted the parameters on which these metrics depend in terms of the two initial conditions q_1 and q_2 of third order. Moreover, we have proven that no further metrics of this kind with $b(t) \neq c(t)$ exist. In the second paper of Kanno, Yasui [24], a power series ansatz for the third and the fourth class of metrics was made. However, our proofs of the smoothness and the convergence of the power series are new.

- (2) In the cases where $N^{k,l}$ is generic or $k = l = 1$ and $a_1(0) = a_2(0) = f(0) = 0$ Assumption 3.11 is satisfied. By calculating W_2^v/W_0^v we

see that the free parameters q , q_1 , q_2 are indeed a subset of the free parameters from Theorem 3.12. Although we do not know if Assumption 3.11 is necessary for Theorem 3.12 to be true, we can make a similar observation in the other two cases from the above theorem. All in all, we hope to have shed some light on the origin of these parameters.

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